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V. A. Sadovnichii

# THEORY OF OPERATORS





# **CONTEMPORARY SOVIET MATHEMATICS**

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# **THEORY OF OPERATORS**

**V. A. Sadovnichii**

*Moscow State University  
Moscow, USSR*

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Roger Cooke**

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## Preface

The second edition of this book contains significant changes and additions intended to bring its content closer to the current curriculum of required courses in functional analysis offered in many universities of the USSR.

A certain part of the material has been rearranged. Appendix I (the theory of measure, measurable functions, and the integral) and Appendix II (distributions and the Fourier transform) have been enlarged and placed in separate chapters.

Chapter 2 on vector spaces has been revised. It now contains a detailed exposition of contemporary material on convex sets in vector spaces and topological vector spaces.

Chapter 4, devoted to the spectral theory of operators, has been enlarged. A section has been added on unbounded operators and the spectral theory of self-adjoint unbounded operators; material on completely continuous operators has also been included. Proofs have been carried out for certain propositions that were contained in the first edition as exercises and more detailed proofs have been given for many propositions.

Chapter 5, The Trace of an Operator, has been enlarged and some new results of the author relating to the traces of discrete operators have been included. These new investigations can be used for finding the regularized traces of partial differential operators. This material can be covered as a separate course. A significant number of examples illustrating the contents have been added, and new exercises have been included to promote better mastery of the material.

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V. A. Lyubishkin, and S. V. Kurochkin, who read the manuscript and made many comments.

V. Sadovnichii



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# Chapter 1

## Metric and Topological Spaces

### 1. ELEMENTARY CONCEPTS OF SET THEORY

#### 1.1. Elementary Properties of Sets. Mappings. Cartesian Product of Sets

A set is a collection of objects having a given property.

Every set is defined by some property  $P$  and consists of those objects and only those objects that have the property.

In what follows we shall agree to consider only sets belonging to some "universal" set  $E$  and to denote the sets under consideration by capital letters  $A, B, C, \dots$  or  $X, Y, Z, \dots$ . The set  $A$  consisting of elements  $x, y, z, \dots$  is often denoted as follows:  $A = \{x, y, z, \dots\}$ .

If the elements  $a$  and  $b$  coincide, we write  $a = b$ . If the elements  $a$  and  $b$  are distinct, we write  $a \neq b$ . The condition that an element  $a$  belongs to the set  $A$  is written as follows:  $a \in A$  and the notation  $a \notin A$  means that the element  $a$  does not belong to the set  $A$  (does not have property  $P$ ).

If it is necessary to emphasize that the set  $A$  is comprised of elements belonging to the universal set  $E$  and having property  $P$ , we often apply the notation

$$A = \{a \in E : P\}.$$

This notation is read as follows: "The set  $A$  consists of the elements of  $E$  having property  $P$ ."

##### 1.1.1. Set Inclusion

Let  $A$  and  $B$  be two sets in  $E$ . The set  $B$  is said to be *contained* in

the set  $A$  (or *included\** in the set  $A$ ) if each element of the set  $B$  is also an element of the set  $A$ . The inclusion of the set  $B$  in the set  $A$  is denoted by the symbol " $\subset$ " and written as follows:  $B \subset A$ . The set  $B$  is not contained in  $A$  ( $B \not\subset A$ ) if there exists at least one element  $b \in B$  such that  $b \notin A$ .

Two sets  $A$  and  $B$  are said to be *coincident* (or *equal*) if they consist of the same elements; in this case we write  $A = B$ .

The inclusion relation of two sets has the following properties:

- 1<sup>0</sup>)  $A \subset A$ ;
- 2<sup>0</sup>) if  $A \subset B$  and  $B \subset A$ , then  $A = B$ ;
- 3<sup>0</sup>) if  $B \subset A$  and  $A \subset C$ , then  $B \subset C$ .

### 1.1.2. The Concept of the Empty Set

Consider the set  $\{a\}$  of elements of  $E$  for which  $a \neq a$ . Such a set does not contain any elements; it is called the *empty set* and is denoted  $\emptyset$ :

$$\emptyset = \{a \in E : a \neq a\}.$$

If a set  $A \neq \emptyset$ , then  $A$  contains at least one element.

The sets  $A$  and  $\emptyset$  are called the *improper* subsets of the set  $A$ . The remaining subsets of  $A$  are called *proper* subsets. The following two properties are obvious:

- 4<sup>0</sup>)  $\emptyset \subset A$  for any  $A$  in  $E$ ;
- 5<sup>0</sup>)  $A \subset E$  for any  $A$  in  $E$ .

### 1.1.3. Operations on Sets

Let  $A$  and  $B$  be two sets in  $E$ . The *union* (or *sum*) of the sets  $A$  and  $B$  is defined to be the set  $C$  consisting of the elements belonging to at least one of the sets  $A$  and  $B$ . The union  $C$  of the two sets  $A$  and  $B$  is denoted as follows:  $C = A \cup B$ .

Similarly  $C = \bigcup_{\alpha} A_{\alpha}$  denotes the union of any number of sets  $A_{\alpha}$ , where the index  $\alpha$  in turn belongs to some set.

The *intersection* of the sets  $A$  and  $B$  is defined to be the set  $C$  consisting of the elements that belong to both the sets  $A$  and  $B$ . The intersection of the two sets  $A$  and  $B$  is denoted as follows:  $C = A \cap B$ .

In exactly the same way  $C = \bigcap_{\alpha} A_{\alpha}$  denotes the intersection of any number of sets  $A_{\alpha}$ .

The operations just introduced have the following properties, whose verification is immediate:

---

\*It is a subset.



- 6<sup>0</sup>)  $A \cup B = B \cup A$  (commutativity of union);  
 7<sup>0</sup>)  $A \cap B = B \cap A$  (commutativity of intersection);  
 8<sup>0</sup>)  $A \cup (B \cap C) = (A \cup B) \cap C$  (associativity of union);  
 9<sup>0</sup>)  $A \cap (B \cup C) = (A \cap B) \cup C$  (associativity of intersection);  
 10<sup>0</sup>)  $A \cup A = A$ ;  
 11<sup>0</sup>)  $A \cap A = A$ ;  
 12<sup>0</sup>)  $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$  (distributivity of intersection), and

$$\left( \bigcup_{\alpha} A_{\alpha} \right) \cap B = \bigcup_{\alpha} (A_{\alpha} \cap B);$$

- 13<sup>0</sup>)  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$  (distributivity of union), and

$$\left( \bigcap_{\alpha} A_{\alpha} \right) \cup C = \bigcap_{\alpha} (A_{\alpha} \cup C);$$

- 14<sup>0</sup>)  $A \cup \emptyset = A$ ;  
 15<sup>0</sup>)  $A \cap \emptyset = \emptyset$ ;  
 16<sup>0</sup>)  $A \cup E = E$ ;  
 17<sup>0</sup>)  $A \cap E = A$ ;  
 18<sup>0</sup>)  $A \subset B$  is equivalent to  $A \cup B = B$  and to  $A \cap B = A$ .

Properties 1<sup>0</sup>)–18<sup>0</sup>) possess duality in the sense that if the symbols  $\subset$ ,  $\cup$ , and  $\emptyset$  are replaced by  $\supset$ ,  $\cap$ , and  $E$  respectively in any one of them, the result is another formula from the same list of 18 formulas. Thus to each theorem whose proof is based on one of the properties 1<sup>0</sup>)–18<sup>0</sup>) there corresponds a dual theorem.

The *difference* of the sets  $A$  and  $B$  is the set of elements of  $A$  that do not belong to  $B$ . The difference of the sets  $A$  and  $B$  is denoted as follows:  $A \setminus B$ . Thus  $A \setminus B = \{x \in E : x \in A \text{ and } x \notin B\}$ . In this definition it is not assumed that  $A \supset B$ .

The *complement* of the set  $A$ , denoted  $A'$ , is defined to be the set of elements of  $E$  not belonging to  $A$ :

$$A' = \{x \in E : x \notin A\} = E \setminus A.$$

The following properties are obvious:

- 19<sup>0</sup>)  $A \cup A' = E$ ;  
 20<sup>0</sup>)  $A \cap A' = \emptyset$ ;  
 21<sup>0</sup>)  $\emptyset' = E$ ;  
 22<sup>0</sup>)  $E' = \emptyset$ ;  
 23<sup>0</sup>)  $(A')' = A$ ;

- 24<sup>0</sup>) The relation  $A \subset B$  is equivalent to  $A' \supset B'$ ;  
 25<sup>0</sup>)  $(A \cup B)' = A' \cap B'$  (the complement of a union is the intersection of the complements),  $\left(\bigcup_{\alpha} A_{\alpha}\right)' = \bigcap_{\alpha} A'_{\alpha}$ ;  
 26<sup>0</sup>)  $(A \cap B)' = A' \cup B'$  (the complement of an intersection is the union of the complements),  $\left(\bigcap_{\alpha} A_{\alpha}\right)' = \bigcup_{\alpha} A'_{\alpha}$ .

Properties 19<sup>0</sup>)–26<sup>0</sup>) also possess duality, just like properties 1<sup>0</sup>)–18<sup>0</sup>).

The *symmetric difference* of two sets  $A$  and  $B$  is the set  $C$  defined as follows:  $C = (A \cup B) \setminus (A \cap B)$ . The symmetric difference of the sets  $A$  and  $B$  is denoted  $A \Delta B$ . It is easy to see that  $A \Delta B = (A \setminus B) \cup (B \setminus A)$ .

#### 1.1.4. Mappings. The Cartesian Product of Sets

The most important concept in analysis is the concept of a mapping of one set into another.

Let  $A$  and  $B$  be two sets. Suppose that with each element  $a$  of the set  $A$  there is associated a definite element  $b = g(a)$  belonging to the set  $B$ . In this case a mapping  $g$  is defined from the set  $A$  into the set  $B$  and can be concisely denoted as follows:

$$g : A \rightarrow B.$$

The element  $b$  is called the *image* of the element  $a$  under the mapping  $g$ , and the element  $a$  is called the *preimage* of the element  $b$ . The element  $a \in A$  is often called a *variable* and the element  $g(a) \in B$  the *value* of  $g$  at the element  $a$ .

If each element  $b$  of the set  $B$  has at least one preimage  $a$  under the mapping  $g$ , we say that the mapping  $g$  is a mapping of  $A$  onto  $B$ .

Let  $M \subset A$ . Then  $g(M)$  denotes the set of those elements of  $B$  that are the images of elements  $a \in M$ . The set  $g(M)$  is called the *image* of the set  $M$  under the mapping  $g$ .

Thus if  $g : A \rightarrow B$  and  $g(A) = B$ , then  $g$  is a mapping of  $A$  onto  $B$ . If  $g(A) \subset B$ , we say that  $g$  is a mapping of  $A$  into  $B$ .

If  $N \subset B$ , we denote by  $g^{-1}(N)$  the set of elements of  $A$  whose images under the mapping  $g$  lie in  $N$ . The set  $g^{-1}(N)$  is called the *complete preimage* of the set  $N$  under the mapping  $g$ .

It is sometimes convenient to call a mapping  $g : A \rightarrow B$  a *function* with domain of definition  $A$  and range of values contained in  $B$ . In some areas of mathematics, depending on the nature of the sets  $A$  and  $B$  and the properties of  $g$ , the mapping  $g$  is called an *operator*, a *functional*, etc.



A mapping  $g$  of the set  $A$  onto the set  $B$  is said to be *one-to-one* (or a *bijection*) if each element of the set  $B$  has only one preimage under the mapping  $g$ .

If  $g : A \rightarrow B$  and if it follows from the relation  $a \neq a'$  that  $g(a) \neq g(a')$ , the mapping  $g$  is called an *injection*. Thus in this case for any  $b \in B$  the equation  $g(a) = b$  has at most one solution. An injection is a one-to-one mapping of  $A$  into  $B$ .

If  $g : A \rightarrow B$  and if for each  $b \in B$  the equation  $g(a) = b$  has at least one solution, then the mapping  $g$  is a *surjection*. A surjection is a mapping of  $A$  onto  $B$ .

According to what has been said a bijection is simultaneously an injection and a surjection, i.e., for any  $b \in B$  the equation  $g(a) = b$  has one and only one solution.

Obviously if  $g$  is a one-to-one mapping of a set  $A$  onto a set  $B$  or a one-to-one correspondence between the elements of these two sets, it is possible to define the mapping  $g^{-1}$  inverse to  $g$ , i.e., knowing the element  $b$  it is possible to determine the element  $a$  uniquely from the equation  $g(a) = b$  and then set  $a = g^{-1}(b)$ .

Let  $A$  be a set. Consider a subset  $R$  of the set of all ordered pairs  $(a, b)$  of elements of this set. If  $(a, b) \in R$ , we say that  $a$  and  $b$  are connected by the relation  $\varphi = \varphi_R$  and denote this fact  $a \underset{\varphi}{\sim} b$ . The relation  $\varphi$  is called an *equivalence relation* if it is *reflexive* (i.e.,  $a \underset{\varphi}{\sim} a$  for any element  $a \in A$ ), *symmetric* (i.e., if  $a \underset{\varphi}{\sim} b$ , then  $b \underset{\varphi}{\sim} a$ ), and *transitive* (i.e., if  $a \underset{\varphi}{\sim} b$  and  $b \underset{\varphi}{\sim} c$ , then  $a \underset{\varphi}{\sim} c$ ).

It is not difficult to verify that these conditions are necessary and sufficient for the relation  $\varphi$  to partition the set  $A$  into disjoint classes.

Indeed, a partition of the set into classes defines a certain equivalence relation. In this situation  $a \underset{\varphi}{\sim} b$  means that  $a$  and  $b$  belong to the same class.

Conversely, if  $\varphi$  is some equivalence relation on the set  $A$  and  $K_a$  is the class of elements  $x \in A$  equivalent to  $a$ , then by reflexivity  $a \in K_a$ . We shall show that two such classes either do not intersect or coincide. Let  $c \in A$  and  $c \in K_a$  and  $c \in K_b$ , i.e.,  $c \underset{\varphi}{\sim} a$ , and  $c \underset{\varphi}{\sim} b$ . Then by symmetry  $a \underset{\varphi}{\sim} c$  and by transitivity  $a \underset{\varphi}{\sim} b$ . By this relation if  $x \in K_a$ , i.e., then  $x \underset{\varphi}{\sim} a \underset{\varphi}{\sim} b$ , and therefore  $x \underset{\varphi}{\sim} b$ , i.e.,  $x \in K_b$ . In exactly the same way it is proved that each element  $y \in K_b$  belongs to  $K_a$ . Thus two classes  $K_a$  and  $K_b$  having a common element must coincide.

If  $g$  is a mapping of the set  $A$  into  $B$ , then the elements of the set  $A$  whose images coincide form disjoint classes in the set  $A$ , i.e., partitioning

into classes is closely connected with the concept of a mapping.

We now pass to the study of an important concept—the Cartesian product of sets. Let  $\Omega = \{1, 2, \dots, n\}$ , and let  $A_1, A_2, \dots, A_n$  be subsets of some set  $A$ . The *Cartesian product* of the sets  $A_k$ , denoted  $\prod_{k=1}^n A_k$ , is defined as the set of functions  $f$  mapping  $\Omega$  into  $A$  such that  $f(k) \in A_k$  for all  $k = 1, \dots, n$ . Obviously  $\prod_{k=1}^n A_k$  can be regarded as the set of all possible collections  $(a_1, a_2, \dots, a_n)$  with  $a_k \in A_k$ . Similarly if  $\Omega = \{1, 2, 3, \dots\}$ , then  $\prod_{k=1}^{\infty} A_k$  is the set of all sequences  $\{a_1, a_2, a_3, \dots\}$ , with  $a_k \in A_k$  for any  $k$ .

In exactly the same way if  $\Omega$  is an arbitrary set and a subset  $A_\alpha$  of the set  $A$  is defined for each  $\alpha \in \Omega$ , the *Cartesian product*  $\prod_{\alpha} A_\alpha$  of the sets  $A_\alpha$  is defined as the set of functions  $f$  mapping  $\Omega$  into  $A$  for which  $f(\alpha) \in A_\alpha$  for all  $\alpha \in \Omega$ .

If  $\Omega = \{1, 2, \dots, n\}$ , then  $\prod_{k=1}^n A_k$  is also denoted  $A_1 \times A_2 \times \dots \times A_n$ ; if  $A = A_i = A_j$  for any  $i, j = 1, \dots, n$ , the notation  $A \times A \times \dots \times A = A^n$  is used.

The concept of the upper limit of a sequence of sets is also of interest. Suppose some infinite sequence of sets  $\{A_n\}$  is given. The set  $A$  consisting of the points belonging to an infinite number of the sets  $A_n$  is called the *upper limit* of the sequence of sets  $A_n$  and denoted as follows:

$$A = \overline{\lim} A_n.$$

The *lower limit* of the sequence of sets  $\{A_n\}$  is defined as the set  $A$  consisting of the elements belonging to all but a finite number of the sets  $A_n$ . For the lower limit of a sequence of sets we use the following notation:

$$A = \underline{\lim} A_n.$$

If a sequence of sets is monotonically increasing, i.e.,  $A_1 \subset A_2 \subset A_3 \subset \dots$ , then

$$\overline{\lim} A_n = \underline{\lim} A_n = \bigcup_{i=1}^{\infty} A_i.$$

Similarly if a sequence of sets is monotonically decreasing, then

$$\overline{\lim} A_n = \underline{\lim} A_n = \bigcap_{i=1}^{\infty} A_i.$$



## 1.2. The Cardinality of a Set

Two sets are said to be *equivalent* if a one-to-one correspondence exists between them. We shall say that equivalent sets have the same *cardinality* or *cardinal number*. Thus with each set a certain object is associated—its cardinality—and the same cardinality is associated with equivalent sets.

A set is called *finite* if it is equivalent to the set of natural numbers  $\{1, 2, \dots, n\}$  for some  $n$ . It is natural to denote the cardinality of such a set by the same letter  $n$ .

The first infinite cardinal is the cardinality of the set of all natural numbers  $\{1, 2, \dots\}$ . Sets of this cardinality are called *countable*, and we shall denote their cardinality by the letter  $\aleph$ .

The cardinality of the set of points of the interval  $[0, 1]$  is called the *cardinality of the continuum*. This cardinality is denoted by the letter  $c$ .

The cardinality of an arbitrary set  $X$  will be denoted by the symbol  $m(X)$ .

### EXAMPLES

1. The set of points of a sphere in three-dimensional space is equivalent to the set of points of the extended plane. A one-to-one correspondence can be established using stereographic projection, for example.

2. The set of rational numbers is countable. Let  $r = p/q$ , where  $q > 0$  and  $p$  and  $q$  are integers and the fraction is in lowest terms. We call the number  $|p| + q$  the *height* of the rational number  $f$ . It is clear that the number of fractions having a given height is finite. It then remains only to enumerate all the rational numbers having heights  $1, 2, \dots$ . Then every rational number will receive one index—a natural number.

3. The set of points of an interval  $[a, b]$ ,  $a \neq b$ , is uncountable. Indeed suppose, to the contrary, that the set of points of the interval can be arranged in a sequence

$$x_1, x_2, \dots, x_n, \dots$$

Divide the interval  $[a, b]$  into three equal parts. Choose a part not containing the point  $x_1$  in either its interior or on its boundary. We denote the interval chosen by  $\lambda_1$ . We then denote by  $\lambda_2$  one of the three equal parts of the interval  $\lambda_1$  not containing the point  $x_2$ , etc. The infinite sequence of intervals  $\lambda_1 \supset \lambda_2 \supset \dots \supset \lambda_n \supset \dots$  has one common point  $\gamma$  by a well-known theorem of analysis. The point  $\gamma$  belongs to each of the intervals  $\lambda_k$  and consequently cannot coincide with any of the points  $x_k$ . Thus the sequence  $x_1, x_2, \dots, x_n, \dots$  cannot contain all the points of an interval.

4. The union of a finite or countable number of countable sets is again a countable set. Every subset of a countable set is either finite or countable. The union of two sets of cardinality of the continuum has cardinality of the continuum. This example illustrates the peculiar arithmetic of cardinal numbers.

Let us prove, for example, that the union of a countable collection of countable sets is a countable set. Let  $A_1, A_2, A_3, \dots$  be a collection of sets, each of which is countable.

Arrange the elements of the sets  $A_1, A_2, A_3, \dots$  as sequences:

$$\begin{aligned} A_1 &= \{a_{11}, a_{12}, a_{13}, \dots\}, \\ A_2 &= \{a_{21}, a_{22}, a_{23}, \dots\}, \\ A_3 &= \{a_{31}, a_{32}, a_{33}, \dots\}, \\ &\vdots \\ A_n &= \{a_{n1}, a_{n2}, a_{n3}, \dots\}, \\ &\vdots \end{aligned}$$

$$\text{Let } A = \bigcup_n A_n = \bigcup_{n=1}^{\infty} A_n.$$

We enumerate the elements  $a$  of the set  $A = \{a\}$  as follows:

$$a_1 = a_{11}, \quad a_2 = a_{21}, \quad a_3 = a_{12}, \quad a_4 = a_{31}, \quad a_5 = a_{22}, \quad a_6 = a_{13},$$

etc.

Some of the sets  $A_i$  and  $A_j$  may have common elements. In this case we count them only once. Thus the elements of the set  $A$  can be enumerated, i.e., placed in one-to-one correspondence with the set of natural numbers  $N$ , so that  $A$  is countable.

5. The closed interval  $[0, 1]$  and the open interval  $(0, 1)$  are equivalent sets.

A one-to-one correspondence can be established, for example, as follows: with the point  $x_n = 1/(n+1)$  of the interval  $(0, 1)$  we associate a point  $y_n$  of the interval  $[0, 1]$ ,  $n = 1, 2, 3, \dots$ . Here  $y_1 = 0$ ,  $y_2 = 1$ , and  $y_k = x_{k-2}$  for  $k = 3, 4, 5, \dots$ . With the remaining points of  $(0, 1)$  we associate the points with the same abscissas.

6. The transcendental numbers form an uncountable set. (A real number is called *transcendental* if it is not a root of any equation of the form  $a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$ ,  $n$  a natural number, and  $a_i$  integers,  $a_0 \neq 0$ ,



$i = 0, 1, \dots, n$ .) Real numbers that are roots of such equations are called *algebraic*.

The set of algebraic numbers, as is easy to show, is countable. Thus, since all the numbers on the line  $\mathbf{R}^1$  form a set of cardinality of the continuum, the set of transcendental numbers is uncountable. In this argument we have made use of an equivalence of the interval  $(0, 1)$  and the line  $\mathbf{R}^1$ :  $y = \frac{1}{\pi} \arctan x + \frac{1}{2}$ , and also the fact that the union of two countable sets is a countable set.

7. The sets of points of an interval and of a square are equivalent.

Let  $I$  be the set of points of the interval  $[0, 1]$  and  $X$  the set of decimal expansions of the numbers (points) of the interval  $I$ . Certain numbers  $t$  have two decimal expansions:  $t^-$  ending in nines and  $t^+$  ending in zeros. (For 1 we fix a unique expansion  $0.99 \dots 9 \dots$ ). The numbers having two expansions form a countable set. We enumerate them:  $t_1, t_2, \dots, t_k, \dots$ , and call them the numbers of first kind, the other numbers being numbers of second kind. We construct a bijection  $\varphi : X \rightarrow I$  by setting  $\varphi(t) = t$  if  $t$  is a number of second kind and  $\varphi(t_i^-) = t_{2i-1}$ ,  $\varphi(t_i^+) = t_{2i}$ .

There exists a unique bijection  $\psi : X \rightarrow X \times X$  defined as follows:

$$\psi(0.\alpha_1\alpha_2\alpha_3\dots\alpha_k\dots) = (0.\alpha_1\alpha_3\dots\alpha_{2k-1}\dots, 0.\alpha_2\alpha_4\dots\alpha_{2k}\dots).$$

We define a mapping

$$\varphi \times \varphi : X \times X \rightarrow I \times I,$$

setting

$$\varphi \times \varphi(a, b) = (\varphi(a), \varphi(b)).$$

Then the composition of the mappings

$$I \xrightarrow{\varphi^{-1}} X \xrightarrow{\psi} X \times X \xrightarrow{\varphi \times \varphi} I \times I$$

is a bijection between the points of the interval and the square.

This result can be stated as follows: the square of a set of cardinality of the continuum again has cardinality of the continuum.

8. Let  $C = A \cup B$  and let  $C$  have cardinality of the continuum. Then either  $A$  or  $B$  has cardinality of the continuum. Indeed  $C$  is equivalent to the square  $[0, 1] \times [0, 1]$ . Now suppose to the contrary that neither  $A$  nor  $B$  has cardinality of the continuum. Consider the vertical intervals whose lower endpoints are points of the interval  $[0, 1]$  of the  $Ox$ -axis and whose upper endpoints lie on the opposite side of the square. Then none of the intervals under consideration can consist entirely of images of points of the

set  $A$  for example, under our bijection of the set  $C$  onto the square. Thus on each such interval there is at least one image from the set  $B$ . This is true for each interval. We have now obtained a contradiction to our earlier assumption.

### 1.3. Partial Ordering. Ordering

Sets situated on the number line are ordered in a natural way, i.e., between any two elements it is possible to place a definite inequality sign. However in many important cases (some of which we shall encounter below) such an "order relation," as it is called, does not always hold on a set. Therefore we shall make the following definitions.

Let the set  $A$  be given. Further suppose a certain subset of  $R$  of the set of all pairs of elements of  $A$  is distinguished, i.e.,  $R \subset A \times A$ . If a pair  $(a, b)$  belongs to  $R$ , we shall write this fact as follows:  $a < b$ . We say that the relation " $<$ " is a *partial ordering* if the following conditions hold:

- 1) Given  $a < b$  and  $b < c$ , it follows that  $a < c$ ;
- 2)  $a < a$  for all  $a \in A$ ;
- 3) given  $a < b$  and  $b < a$ , it follows that  $a = b$ .

(Properties 2) and 3) show that the ordering is not strict, i.e., does not exclude the possibility that the two elements are the same.) Elements  $a$  and  $b$  for which one of the relations  $a < b$  or  $b < a$  holds are said to be *comparable*, and the initial set  $A$  is said to be *partially ordered* by the relation  $<$ .

If it is known that for any two distinct elements  $a$  and  $b$  of the set  $A$  either  $a < b$  or  $b < a$ , the set  $A$  is said to be *ordered* by the relation  $<$ .

A subset  $B$  of a set  $A$  partially ordered by the relation  $<$  is said to be *bounded above* if there exists an element  $a \in A$  such that  $b < a$  for any  $b \in B$ . Any such element  $a$  is called an *upper bound* of the set  $B$ . (A lower bound is defined similarly.) If in addition  $a < c$  for every other upper bound  $c$  of the set  $B$ , then  $A$  is called the *least upper bound* or *supremum* of the set  $B$ . (The greatest lower bound, or *infimum*, is defined similarly.)

If some element  $m$  of a partially ordered set  $A$  possesses the property that the relations  $p \in A$  and  $m < p$  imply  $p = m$ , then  $m$  is called a *maximal element* of  $A$ . (A *minimal element* is defined similarly.)

We shall often have to deal with objects that are infinite when regarded as sets. In proving theorems about such objects the following lemma is often used.

**ZORN'S LEMMA.** *If every ordered subset  $B$  of a nonempty partially ordered set  $A$  has a least upper bound, then  $A$  has a maximal element.*

A nonempty ordered set is said to be *well-ordered* if any nonempty subset of it has a minimal element.



**ZERMELO'S THEOREM.** *Every set can be well-ordered by introducing some order relation.*

The proof of Zermelo's Theorem is based on the so-called axiom of choice, which asserts that if any system of nonempty pairwise disjoint sets is given there exists a new set containing exactly one element from each of the sets of the system.

The assertion of the axiom of choice seems intuitively clear, but the use of this axiom leads to nonconstructive proofs, since the rule for making a choice cannot be exhibited explicitly. Many facts established using the axiom of choice are not intuitive. (For example, one can decompose a ball into four equal pieces in such a way that from two of the pieces one can form a whole ball of the same radius by "moving" them as "rigid" bodies. From the other two pieces one can form an identical ball.)

It can be shown that Zorn's Lemma, the axiom of choice, and Zermelo's Theorem are equivalent assertions. They are a generalization of the principle of mathematical induction to the case of uncountable sets.

In some areas of functional analysis one uses the concept of a directed set.

A partially ordered set  $A$  is said to be *directed* if every finite subset of it has an upper bound.

It is easy to verify that for a set to be directed it suffices that every two-element subset of it have an upper bound.

#### 1.4. Comparison of Cardinalities

Let  $A$  and  $B$  be two arbitrary sets. If  $A$  is equivalent to  $B$ , then their cardinalities (by definition) are equal. If one of the sets, for example  $A$ , is equivalent to some subset of the set  $B$ , we say that the cardinality of  $A$  *does not exceed* the cardinality of the set  $B$ , and we write:  $m(A) \leq m(B)$ . If in addition there is no subset of  $A$  equivalent to  $B$ , then it is natural to say that the cardinality of  $A$  is *less than* that of  $B$ , and write  $m(A) < m(B)$ . Theoretically two other cases could occur:

1.  $B$  contains a set equivalent to  $A$  and  $A$  contains a set equivalent to  $B$ .
2. The sets  $A$  and  $B$  are not equivalent, and neither contains a subset equivalent to the other.

In the first case one can show that the sets  $A$  and  $B$  are equivalent. As for the second case, it is actually impossible. This can be deduced from Zermelo's Theorem. Thus the cardinalities of any two sets  $A$  and  $B$  are comparable. The order relation thus introduced satisfies properties 1)–3) of Section 3. The countable cardinality is the smallest infinite cardinality. The question whether the cardinality of the continuum is the next cardinality

after the countable cardinality or there are others between them (the so-called continuum hypothesis) resisted efforts to answer it for a long time. It has recently been shown that the assertion that there are no intermediate cardinalities does not contradict the other axioms of set theory and cannot be deduced from these axioms.

### EXAMPLES

1. Let  $A$  be some nonempty set and  $M = \{B\} \stackrel{\text{def}}{=} 2^A$  the set of all subsets  $B$  of  $A$ .

We shall assume that  $B_1 < B_2$  means  $B_1 \subset B_2$ . Obviously this relation is an order relation satisfying 1)–3) of Section 3. It is also clear that in the general case  $M$  is not ordered (or well-ordered).

If  $N$  is any subset of the set  $M$ , it is bounded above. Its least upper bound is the set  $\tilde{N} = \bigcup_{B \in N} B$ . There exists a maximal element in  $M$ : it is the set  $A$  itself regarded as a subset, and the assertion of Zorn's Lemma is obvious. Zermelo's Theorem asserts that  $M$  can be well-ordered. However, it is not clear from the theorem how to do this.

2. If some set  $A$  is given, the set  $M$  whose elements are all the subsets of the set  $A$  has cardinality larger than  $A$ .

Indeed, denote the cardinality of the set  $A$  by  $m(A)$ , and the cardinality of the set  $M$  by  $m(M) = 2^{m(A)}$ . It is obvious that  $m(M) \geq m(A)$ . We shall exclude the possibility of equality  $m(M) = m(A)$ . Assuming the contrary, we establish a one-to-one correspondence between the elements  $\{a\}$  of the set  $A$  and the elements  $\{B\}$  of the set  $M$ —subsets of the set  $A$ . We form the set  $B_0$  consisting of all elements  $\{a\}$  not belonging to the sets to which they correspond under the one-to-one mapping of  $A$  onto  $M$ . Let  $a_0$  be the element of  $A$  corresponding to  $B_0$ . The element  $a_0$  cannot belong to the set  $B_0$ , and yet cannot fail to belong to it. We have thus obtained a contradiction.

A set containing  $n$  elements obviously has  $2^n$  subsets. Let us verify that the set of subsets of a countable set has cardinality of the continuum. Indeed each point of the interval has an expansion into a binary fraction. Each such expansion can be interpreted as a subset of the set of natural numbers (a number  $n$  belongs to the set or not according as the integer 1 or 0 stands in the  $n$ th place). To each point of the interval there corresponds at least one and at most two expansions.\* We obtain the result that

$$c \leq 2^{\aleph_0} \leq 2 \cdot c$$

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\*Cf. Section 2.3 of this chapter for more details.



(where  $c$  is the cardinality of the continuum and  $a$  is the cardinality of a countable set). But (cf. Example 4 of Section 2)  $c = 2 \cdot c$ , and so  $2^a = c$ .

This example shows in particular that the set of points of an interval is uncountable, i.e. the cardinality of the continuum is not countable, but is indeed a new cardinality. We see also that "infinite" is not simply a contrary to "finite": there exist many infinite sets not equivalent to one another.

In particular the set of all subsets of a set of cardinality of the continuum has cardinality larger than the cardinality of the continuum—the so-called cardinality of the hypercontinuum.

3. If  $A = \overline{\lim_{n \rightarrow \infty} A_n}$ , then  $A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$ .

The proof is obtained by standard reasoning: Let  $a \in A$ ; then  $a \in \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$ , and conversely.

4. A family  $\{\Sigma\}$  of subsets of some set is called a *filter* if:

- a)  $\emptyset \notin \{\Sigma\}$ ;
- b) if  $A \supset B$  and  $B \in \{\Sigma\}$ , then  $A \in \{\Sigma\}$ ;
- c) if  $A, B \in \{\Sigma\}$ , then  $A \cap B \in \{\Sigma\}$ .

If  $\{\Sigma\}$  and  $\{F\}$  are two filters and  $\{\Sigma\} \supset \{F\}$ , we say that the filter  $\{\Sigma\}$  *majorizes*  $\{F\}$ . An *ultrafilter* is a filter that is not majorized by any filter except itself. It follows from Zorn's Lemma that every filter is majorized by some ultrafilter.

## 2. METRIC SPACES

A very important role in mathematical analysis is played by the concept of limit. The basis of each of the various definitions of limit is some concept of closeness between objects. It is therefore natural to try to introduce a concept of distance between elements for sets of an arbitrary nature, and then the concept of passage to a limit. Moreover we shall see that many properties of a metric space depend only on the collection of its so-called open subsets. The concept of an open set in turn can be taken as the basis for the definition of even more general spaces—topological spaces.

### 2.1. Definition of a Metric Space. Examples

DEFINITION 1. The structure of a *metric space* is defined on the set  $X$  if a function of a pair of arguments is defined

$$\rho : X \times X \rightarrow \mathbf{R}^1, \quad \mathbf{R}^1 \text{ is the real line,}$$

possessing the following properties:



- 1)  $\rho(x, y) = 0$  if and only if  $x = y$ ;
- 2)  $\rho(x, y) = \rho(y, x)$  (the property of symmetry);
- 3)  $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$  (the triangle inequality).

The function  $\rho(x, y)$ ,  $x, y \in X$ , is called a *metric* or *distance function*, and the number  $\rho(x, y)$  is called the *distance* between the points  $x$  and  $y$ .

Thus the pair consisting of the set  $X$  and the function  $\rho$  form a metric space; we shall denote it by

$$(X, \rho) \quad \text{or} \quad R = (X, \rho),$$

or simply by  $X$  if it is clear which metric is meant.

If we set  $x = y$  in 3), taking account of 1) and 2), we obtain the inequality  $0 \leq \rho(y, z)$ , i.e., the distance function is a nonnegative function of its arguments.

We now give examples of the most common metric spaces.

#### EXAMPLES

1. Arithmetic  $n$ -dimensional space  $X$ , whose points are ordered collections of  $n$  real numbers,  $x = (x_1, \dots, x_n)$ , is a metric space if we set\*

$$\rho(x, y) = \left[ \sum_{i=1}^n |x_i - y_i|^2 \right]^{1/2}.$$

The proof of the triangle inequality for this space is given below in Example 3. In what follows we shall also denote this pair  $(X, \rho)$  by  $\mathbf{R}^n$ .

It is possible to introduce other distance functions in arithmetic  $n$ -space  $X$ , for example:

- a)  $\rho_0(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}$ ;
- b)  $\rho_1(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$ ;
- c)  $\rho_2(x, y) = \sum_{i=1}^n |x_i - y_i|$ ;
- d)  $\rho_3(x, y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y; \end{cases}$
- e)  $\rho_4(x, y) = \begin{cases} \rho(x, y), & \text{if } \rho(x, y) < 1, \\ 1, & \text{if } \rho(x, y) \geq 1. \end{cases}$

Naturally the same set becomes different metric spaces when this is done.

---

\*Cf. also Example 3.

2. Let  $Y$  be the space of continuous functions defined on an interval  $[a, b]$ . We introduce a metric by setting  $\rho(x, y) = \max_{a \leq t \leq b} |x(t) - y(t)|$ . The space  $(Y, \rho)$  so obtained is a metric space. It is denoted by  $C[a, b]$ .

The set of continuous functions can be made into other metric spaces by introducing a distance function, for example, according to the rules a), d), and e) of Example 1 or setting

$$\rho(x, y) = \int_a^b |x(t) - y(t)| dt;$$

or

$$\rho(x, y) = \left[ \int_a^b |x(t) - y(t)|^p dt \right]^{1/p}, \quad p > 1.$$

In exactly the same way the set  $Z$  of  $n$  times continuously differentiable functions on the interval  $[a, b]$ ,  $n \geq 1$ , becomes a metric space if we introduce a metric according to the rule

$$\rho(x, y) = \max_{0 \leq i \leq n} \max_{a \leq t \leq b} |x^{(i)}(t) - y^{(i)}(t)|,$$

$$x^{(0)}(t) \equiv x(t), \quad y^{(0)}(t) \equiv y(t).$$

This space is usually denoted as follows:  $C^n[a, b]$ ,  $n \geq 1$ .

3. Let  $U$  be the set consisting of the sequences of complex numbers  $(x_1, x_2, \dots, x_n, \dots)$  such that  $\sum_{k=1}^{\infty} |x_k|^2 < \infty$ . We introduce a distance function  $\rho(x, y)$  according to the rule

$$\rho(x, y) = \left[ \sum_{i=1}^{\infty} |x_i - y_i|^2 \right]^{1/2},$$

where  $x = (x_1, x_2, \dots)$  and  $y = (y_1, y_2, \dots)$ .

The triangle inequality is obtained by passing to the limit in the following inequality (cf. also Example 1).

$$\left[ \sum_{k=1}^n |z_k - x_k|^2 \right]^{1/2} \leq \left[ \sum_{k=1}^n |z_k - y_k|^2 \right]^{1/2} + \left[ \sum_{k=1}^n |y_k - x_k|^2 \right]^{1/2},$$

This inequality is a consequence of the well-known *Cauchy-Bunyakovskii inequality*.\*

$$\left| \sum_{k=1}^n a_k \bar{b}_k \right|^2 \leq \sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2,$$

---

\*Usually known in the West as the Cauchy-Schwarz inequality. *Tr.*

where  $a_k$  and  $b_k$ ,  $k = 1, \dots, n$  are arbitrary numbers.

The Cauchy-Bunyakovskii inequality can be proved as follows: Let  $\lambda$  be an arbitrary number and  $a$  and  $b$  vectors,  $a = (a_1, \dots, a_n)$ ,  $b = (b_1, \dots, b_n)$ . Then  $(\lambda a - b, \lambda a - b) \geq 0$ , i.e.,

$$\lambda \bar{\lambda}(a, a) - \lambda(a, b) - \bar{\lambda}(b, a) + (b, b) \geq 0.$$

Set  $\lambda = re^{-i\varphi}$ , where  $\varphi = \arg(a, b)$ . We obtain the result that  $r^2(a, a) - 2r|(a, b)| + (b, b) \geq 0$ . Then the discriminant of this quadratic trinomial must be nonpositive, and so the Cauchy-Bunyakovskii inequality is proved.

For the function  $\rho(x, y)$  it is also not difficult to verify the remaining distance axioms.

We remark that if  $\sum_{k=1}^{\infty} |x_k|^2 < \infty$  and  $\sum_{k=1}^{\infty} |y_k|^2 < \infty$ , then  $\sum_{k=1}^{\infty} |x_k \pm y_k|^2 < \infty$  also, and the distance function  $\rho(x, y)$  can really be introduced using the rule given above.

Thus we have defined a metric space  $(U, \rho)$ . It is usually denoted  $l^p$ .

In this same space  $U$  whose points are sequences such that the series of squares of the terms converges, one can introduce a distance function in many ways. One of the interesting ways is as follows: The distance is defined by the formula  $\rho(x, y) = \sup_k |x_k - y_k|$ . It is easy to verify that this upper bound exists and that the function  $\rho(x, y)$  defines a distance.

4. Let  $V$  be the set of sequences of numbers  $(x_1, x_2, \dots, x_n, \dots)$  such that  $\sum_{k=1}^{\infty} |x_k|^p < \infty$ ,  $p \geq 1$ . We define a distance function by the formula

$$\rho(x, y) = \left[ \sum_{k=1}^{\infty} |x_k - y_k|^p \right]^{1/p},$$

where  $x = (x_1, x_2, \dots, x_n, \dots)$  and  $y = (y_1, y_2, \dots, y_n, \dots)$ .

This metric space is denoted  $l^p$ ,  $p \geq 1$ .

The only metric axiom that needs to be verified is the triangle inequality. It can be proved by the following scheme:

a) Suppose  $0 < \alpha < 1$ . Then the inequality

$$f(x) = x^\alpha - \alpha x + \alpha - 1 \leq 0$$

holds for  $x > 0$ .

b) Let  $a > 0$ ,  $b > 0$ ,  $p > 1$ , and  $q$  be such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$



Then, substituting  $x = a/b$  and  $\alpha = 1/p$ , we can obtain\*

$$a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q}.$$

c) Let  $x_i \geq 0$  and  $y_i \geq 0$  for  $i = 1, \dots, n$ . Substituting

$$a = \frac{x_i^p}{\sum_{j=1}^n x_j^p}, \quad b = \frac{y_i^q}{\sum_{j=1}^n y_j^q},$$

we obtain

$$\frac{x_i y_i}{\left(\sum_{j=1}^n x_j^p\right)^{1/p} \left(\sum_{j=1}^n y_j^q\right)^{1/q}} \leq \frac{1}{p} \frac{x_i^p}{\sum_{j=1}^n x_j^p} + \frac{1}{q} \frac{y_i^q}{\sum_{j=1}^n y_j^q}.$$

Summing on  $i$  from 1 to  $n$ , we obtain the inequality

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^p\right)^{1/p} \left(\sum_{i=1}^n y_i^q\right)^{1/q},$$

which is called *Hölder's inequality*.

d) Let  $x_i, y_i, p$  and  $q$  be as above. We write the identity

$$\sum_{i=1}^n (x_i + y_i)^p = \sum_{i=1}^n x_i (x_i + y_i)^{p-1} + \sum_{i=1}^n y_i (x_i + y_i)^{p-1}.$$

Applying Hölder's inequality to each term on the right-hand side and simplifying, we obtain the inequality

$$\left(\sum_{i=1}^n (x_i + y_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^n x_i^p\right)^{1/p} + \left(\sum_{i=1}^n y_i^p\right)^{1/p}.$$

e) Let  $x_i$  and  $y_i, i = 1, \dots, n$ , be complex numbers. Then a consequence of the preceding inequality is *Minkowski's inequality*:

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |y_i|^p\right)^{1/p}.$$

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\*This inequality is called *Young's inequality*.

From Minkowski's inequality it is easy to obtain the triangle inequality for  $p > 1$ . The case  $p = 1$  is immediate.

5. Let  $W$  be the set of sequences  $x = (x_1, x_2, \dots, x_n, \dots)$  of numbers  $x_k$  such that  $\sup_{1 \leq k < \infty} |x_k| < \infty$ . Let  $\rho(x, y) = \sup_{1 \leq k < \infty} |x_k - y_k|$ . Then  $(W, \rho)$  is a metric space. It is obvious that the distance axioms hold here. This metric space is denoted by the symbol  $m$ .

6. Let  $S$  be the set of sequences  $(n_1, n_2, \dots)$  of natural numbers. We define the distance between two such sequences  $x$  and  $y$  by the rule  $\rho(x, y) = 1/k$ , where  $k$  is the first index for which the coordinate  $n_k$  of the sequence  $x$  differs from the corresponding coordinate of the sequence  $y$ . We set  $\rho(x, y) = 0$  if  $x = y$ . Then  $(S, \rho)$  is a metric space known as the Baire zero-dimensional space and usually denoted  $B_0$ .

We remark that, as we have emphasized several times, if  $\rho(x, y)$  is a distance function in some metric space, then we can obtain a new distance function by formulas a) or e) of Example 1.

## 2.2. Open and Closed Sets

DEFINITION 1. The ball  $O(a, r)$  in the space  $X$  [resp. closed ball  $K(a, r)$ ] with center at the point  $a$  and radius  $r$  is defined to be the set of points  $x \in X$  such that  $\rho(x, a) < r$  [resp.  $\rho(x, a) \leq r$ ].

DEFINITION 2. A set  $\Sigma \subset X$  is called *open* in  $X$  if for each of its points  $x$  it contains some ball  $O(x, r)$ .\*

DEFINITION 3. A *neighborhood* of the point  $x \in X$  is defined to be any open set containing  $x$ . A neighborhood of a subset of  $X$ , possibly  $X$  itself, is any open set containing the given subset. We shall denote a neighborhood of the point  $x$  by  $\Sigma_x$ .

DEFINITION 4. Let  $Y \subset X$ . Then a point  $x \in X$  is called a *limit point* of the set  $Y$  if each neighborhood of the point  $x$  contains at least one point  $y$  such that  $y \in Y$  and  $y \neq x$ .

A point  $y \in Y$  is called an *isolated point* of the set  $Y$  if there exists a neighborhood of the point  $y$  in which there are no points of  $Y$  different from  $y$ .

DEFINITION 5. A point  $y \in Y \subset X$  is called *interior* if it is contained in  $Y$  along with some neighborhood of it.\*\* The points interior to the

\*Obviously any open ball in a metric space is an open set (cf. Example 3 at the end of this section).

\*\*The set of all interior points of the set  $Y$  is called the *interior* of  $Y$  and is denoted by  $\overset{\circ}{Y}$ .



complement of  $Y$  in  $X$  are said to be *exterior* in relation to  $Y$ . If a point is neither interior nor exterior in relation to  $Y$ , it is called a *boundary point* of  $Y$ . The set of boundary points of  $Y$  is denoted by  $\partial Y$ .

DEFINITION 6. A set in a metric space is called *closed* if its complement is open.

The following lemma holds.

LEMMA 1. *The union of any number of open sets and the intersection of any finite number of open sets is open; the sets  $\emptyset$  and  $X$  are open.*

*The intersection of any number of closed sets and the union of any finite number of closed sets is closed; the sets  $\emptyset$  and  $X$  are closed.*

PROOF: Let  $\{\Sigma_\alpha\}$  be a family of open sets in  $X$ . If  $x \in \bigcup_\alpha \Sigma_\alpha$ , there exists an index  $\alpha_0$  such that  $x \in \Sigma_{\alpha_0}$ . Then there exists a number  $r > 0$  such that  $O(x, r) \subset \Sigma_{\alpha_0}$ , i.e.,  $O(x, r) \subset \bigcup_\alpha \Sigma_\alpha$ . Further, if  $\Sigma_1, \dots, \Sigma_n$  are open

in  $X$ , the relation  $x \in \bigcap_{i=1}^n \Sigma_i$  implies that  $x \in \Sigma_i$  for any  $i = 1, \dots, n$ , i.e., for  $i = 1, \dots, n$  there exist numbers  $r_i > 0$  such that  $O(x, r_i) \subset \Sigma_i$ . Taking  $r = \min_{1 \leq i \leq n} r_i$ , we obtain the inclusion  $O(x, r) \subset O(x, r_i)$  for any  $i = 1, \dots, n$ ,

i.e.,  $O(x, r) \subset \bigcap_{i=1}^n \Sigma_i$ .

The second assertion follows immediately from the first if we use the duality principle for sets. The fact that  $\emptyset$  and  $X$  are simultaneously open and closed is obvious. ■

DEFINITION 7. The *closure*  $\bar{Y}$  of a set  $Y$  is the intersection of all closed sets containing  $Y$ . It is obvious that  $\bar{Y}$  is contained in every closed set containing  $Y$ . Consequently the closure of  $Y$  is the smallest closed set containing  $Y$ .

The following lemma holds.

LEMMA 2. *The operation of closure in a metric space possesses the following properties:*

$$1) \bar{A} \supset A; \quad 2) \bar{\bar{A}} = \bar{A}; \quad 3) \overline{A \cup B} = \bar{A} \cup \bar{B}; \quad 4) \bar{\emptyset} = \emptyset, \quad \bar{X} = X.$$

PROOF: Property 1) is obvious. If  $x \in A$ , then  $x$  belongs to any closed set containing  $A$ , i.e.,  $x \in \bar{A}$ . Property 2) follows from the fact that  $\bar{A}$  is closed (Lemma 1). Let us prove property 3). The set  $A \cup B$  contains  $A$ ,



whence  $\overline{A \cup B} \supset A$ . (Since every closed set containing  $A \cup B$  also contains  $A$ , the intersection of all such sets also contains  $A$ .) Consequently  $\overline{A \cup B} \supset \bar{A}$ . Similarly  $\overline{A \cup B} \supset \bar{B}$ , and therefore  $\overline{A \cup B} \supset \bar{A} \cup \bar{B}$ . Conversely  $\bar{A} \cup \bar{B}$ , according to what has been proved (Lemma 1) is closed; hence  $\bar{A} \cup \bar{B} \supset \overline{A \cup B}$ . Assertion 4) means that  $\emptyset$  and  $X$  are closed sets. ■

In the course of the preceding proof we have proved that if  $C \subset D$ , then  $\bar{C} \subset \bar{D}$ .

The following proposition holds.

**PROPOSITION 1.** *A point  $a$  belongs to  $\bar{A}$  if and only if each neighborhood  $\Sigma_a$  of the point  $a$  intersects  $A$ .*

**PROOF:** If we assume that  $a \notin \bar{A}$ , then there exists a neighborhood  $\Sigma_a$  (for example, the complement of  $\bar{A}$ ) that does not intersect  $A$  (or even  $\bar{A}$ ), contrary to the hypothesis of the proposition. Conversely let  $a \in \bar{A}$  and let  $\Sigma_a$  be a neighborhood of the point  $a$  that does not intersect  $A$ . Then the complement of  $\Sigma_a$  is a closed set containing  $A$ , hence also containing  $\bar{A}$ , contradicting the fact that  $a \in \bar{A}$ . ■

For a set  $A$  in a metric space we denote by  $\tilde{A}$  the set of limit points of  $A$ . The following proposition holds.

**PROPOSITION 2.** *For any set  $A$  the relations  $\bar{A} = A \cup \tilde{A} = A \cup \partial A$  hold.*

**PROOF:** it is obvious that  $A \subset \bar{A}$ . It follows from Definition 4 and Proposition 1 that  $\tilde{A} \subset \bar{A}$ . If  $a \in \bar{A}$ , then either  $a \in A$  or  $a \notin A$ . In the latter case each neighborhood of the point  $a$  contains a point of  $A$  distinct from  $a$ , i.e.,  $a \in \tilde{A}$ . Thus  $\bar{A} = A \cup \tilde{A}$ . To prove the equality  $\bar{A} = A \cup \partial A$  it suffices to remark that the exterior points of the set  $A$  constitute precisely the complement of the set  $\bar{A}$ . ■

Let  $(X, \rho)$  be a metric space and  $Y$  a subset of  $X$ . The metric  $\rho$  can be restricted to the points of  $Y$ . Thereby  $Y$  itself becomes a metric space and the pair  $(Y, \rho)$  is called a *subspace* of the space  $(X, \rho)$ .

**DEFINITION 8.** Let  $X$  be a metric space and  $Y$  a subspace of  $X$ . A set  $\Sigma_Y \subset Y$  is called *open relative to  $Y$*  if there exists an open set  $\Sigma_X$  in  $X$  such that  $\Sigma_Y = Y \cap \Sigma_X$ .

We shall prove that a set  $\Sigma_Y \subset Y$  is open relative to  $Y$  if and only if it is open in  $Y$ , regarded as a subspace. Let  $\Sigma_Y$  be open relative to  $Y$ , i.e.,  $\Sigma_Y = Y \cap \Sigma_X$ , where  $\Sigma_X$  is open in  $X$ . Then for each point  $y \in Y \cap \Sigma_X$  there exists a ball  $O(y, r)$  contained in  $\Sigma_X$ . The set  $O(y, r) \cap Y$  is then a ball in  $Y$  contained in  $\Sigma_Y$ , i.e.,  $\Sigma_Y$  is open in  $Y$ . Conversely, let  $\Sigma_Y$  be open

in  $Y$ . This means that for each point  $y \in \Sigma_Y$  there exists a ball in  $Y$  with center at  $y$  contained in  $\Sigma_Y$ . For each such ball consider the corresponding ball in  $X$  with the same center and radius. The union of all such balls (over all  $y \in \Sigma_Y$ ) gives us an open set  $\Sigma_X$ . It is obvious that  $\Sigma_Y = Y \cap \Sigma_X$ .

Sets  $F_Y \subset Y$  that are *closed relative to  $Y$*  are defined similarly; a proposition similar to the one given above holds for them.

We emphasize that when we talk about a relatively open (or closed) set, we must indicate along with the space  $X$  the subspace  $Y$  of it relative to which the definitions are given.

For example the interval  $I = (0, 2)$  is not an open set in  $\mathbf{R}^2$ , but is open relative to  $\mathbf{R}^1$  in  $\mathbf{R}^2$ , since  $I = \mathbf{R}^1 \cap O(a, 1)$ , where  $O(a, 1)$  is the open disk in  $\mathbf{R}^2$  with center at the point  $a = (1, 0)$  and radius 1.

**DEFINITION 9.** A space  $X$  is said to be *connected* if it cannot be represented as the union of two nonempty closed (or two nonempty open) disjoint subsets.

A set  $Y$  in a metric space  $X$  is called *connected* if  $Y$  is connected as a subspace of  $X$ .

### 2.3. Everywhere Dense and Perfect Sets

**DEFINITION 10.** Let  $A$  and  $B$  be two sets in a metric space  $X$ . The set  $A$  is called *dense* in  $B$  if  $\bar{A} \supset B$ . The set  $A$  is *everywhere dense* in  $X$  if  $\bar{A} = X$ .

Spaces in which there are countable everywhere dense sets are called *separable*.

It is easy to verify that Examples 1–4 of metric spaces considered above are separable metric spaces. Thus, in  $\mathbf{R}^n$  the set of points for which all coordinates are rational numbers is a countable everywhere dense set. In the spaces  $C[a, b]$  and  $C^n[a, b]$  the set of polynomials with rational coefficients is such a set; in the spaces  $l^2$  and  $l^p$  the set of sequences of rational numbers only a finite number of which are nonzero (the number of nonzero terms is not the same for all sequences in the set) is such a set.

The space  $m$  is an example of a nonseparable space. If we consider the set  $E_0$  of sequences consisting of zeros and ones only, the cardinality of this set is that of the continuum, since in dyadic notation these sequences represent all the numbers of the interval  $[0, 1]$ .

To verify this, we proceed as follows. Let  $x \in [0, 1]$  and let the interval  $[0, 1]$  be divided into two equal parts. After a zero followed by a decimal point place 0 or 1 according as the number  $x$  belongs to the first or the second half. If it belongs to both intervals ( $x = 1/2$ ), one may write either 0 or 1 arbitrarily. The process is repeated indefinitely with the smaller interval



to which the point  $x$  belongs. As a result we obtain a sequence of zeros and ones:  $0.0110001\dots$ . If  $x \neq y$ , then as a result of the divisions these points will belong to different intervals at some stage and therefore the sequences corresponding to them will be different. Thus the set of all sequences of zeros and ones has cardinality not less than that of the continuum. This is sufficient for our purposes. (It is easy to verify that in fact the set of all sequences of zeros and ones is a set of cardinality equal to that of the continuum.) The mutual distances between any two different elements  $x$  and  $y$  of the set  $E_0$  are all equal to 1. Therefore it is impossible to approximate arbitrarily closely each of these points by elements of a countable set, since the set of balls with centers at points of the set  $E_0$  and radius  $1/3$  is a set of cardinality of the continuum and these balls do not intersect. Since  $E_0 \subset m$ , the space  $m$  is nonseparable.

A set  $A$  is called *nowhere dense* in a metric space  $R$  if any open set of this space contains an open subset disjoint from the set  $A$ .

For example in the space  $C[0,1]$  the set  $A$  of functions of the form  $y = nx^2$  ( $n$  integers) is nowhere dense. Another example of a nowhere dense set on the interval  $[0,1]$  (regarded as a metric space) is given by the so-called "Cantor perfect set."

A set  $A$  situated in a metric space is called *perfect* if it is closed and each point of the set  $A$  is a limit point of  $A$ .

The Cantor perfect set on the interval  $I = [0,1]$  is constructed as follows. The interval  $(1/3, 2/3)$  is removed from the interval  $[0,1]$ . The remaining set—the union of the two intervals  $[0, 1/3]$  and  $[2/3, 1]$ —is denoted by  $I_1$ . A third of each of these two intervals in turn is removed: the intervals  $(1/9, 2/9)$  and  $(7/9, 8/9)$ . The union of the remaining intervals is denoted by  $I_2$ . We continue this process indefinitely. It is obvious that  $I \supset I_1 \supset I_2 \supset \dots$  and  $I_n$  is the union of  $2^n$  intervals, each of length  $3^{-n}$ .

The set  $K = \bigcap_{n=1}^{\infty} I_n$  is called the *Cantor set*.

We shall show that  $K$  is perfect. The fact that it is closed follows from the construction and Lemma 1. It remains to show that  $K$  contains no isolated points. Let  $x \in K$  and let  $\Sigma_x$  be an arbitrary neighborhood of the point  $x$ . Then by definition of an open set there exists an interval  $\sigma_x$  (a ball with center at the point  $x$ ) containing the point  $x$  and  $\sigma_x \subset \Sigma_x$ . Let  $\Lambda_n$  be the interval of the set  $I_n$  that contains the point  $x$ . If  $n$  is sufficiently large, then  $\Lambda_n \subset \sigma_x$ . We denote by  $a_n$  an endpoint of the interval  $\Lambda_n$  that does not coincide with  $x$ . It follows from the construction of the set  $K$  that  $a_n \in K$ . Consequently an arbitrary neighborhood of the point  $x$ —the set  $\Sigma_x$ —contains a point  $a_n \neq x$ :  $a_n \in \Lambda_n \subset \sigma_x \subset \Sigma_x$ , i.e., the point  $x$  is a limit point for the set  $K$  and consequently  $K$  is perfect.



We now prove that  $K$  is nowhere dense on the interval  $[0, 1]$  regarded as a metric space with the usual Euclidean distance. Since any open set on the interval contains an interval, it suffices to show that any interval (ball) contains another interval disjoint from the set  $K$ . Let  $\sigma$  be an arbitrary interval of the segment  $[0, 1]$ . If it contains no points of the set  $K$ , the construction is finished. But if there is a point  $x \in K$  and  $x \in \sigma$ , we may choose a natural number  $m$  so large that  $x \in \Lambda_m \subset I_m$  and  $\Lambda_m \subset \sigma$ . Choose an interval of length  $1/3^{m+1}$  with center at the midpoint of  $\Lambda_m$ . This interval contains no points of the set  $K$  and is contained in  $\sigma$ .

Thus the set  $K$  is nowhere dense on the interval  $[0, 1]$ .

REMARK. It is easy to see that if a set is closed and is not nowhere dense, then it contains some ball.

Indeed, suppose the contrary, i.e., that there is no ball consisting entirely of points of the given closed set. That is, no matter what ball we choose, it always contains a point of the complement of the given set. Since the complement is open, each point of the complement is surrounded by a ball made up of points of the complement. Thus we see that each ball contains a ball disjoint from the given set, i.e., the set is nowhere dense.

We have now reached a contradiction.

## 2.4. Convergence, Continuous Mappings

DEFINITION 11. A sequence  $\{a_n\}$  of points of a metric space is said to *converge* to the point  $a$  of this space if any neighborhood of the point  $a$  contains all but a finite number of terms of the sequence. If the sequence  $a_n$  converges to  $a$ , we write  $a_n \rightarrow a$  as  $n \rightarrow \infty$  or  $\lim_{n \rightarrow \infty} a_n = a$ . It follows directly from the definition just given that if  $a_n \rightarrow a$ , then  $\rho(a_n, a) \rightarrow 0$  as  $n \rightarrow \infty$ .

The following proposition holds.

LEMMA 3. A point  $a \in R$  belongs to the closure  $\bar{A}$  of some set  $A$  if and only if there exists a sequence  $\{a_n\}$  of points of  $A$  converging to  $a$ .

PROOF: If  $a_n \rightarrow a$ , then each neighborhood of the point  $a$  contains points from  $\{a_n\}$ , and so intersects  $A$ , i.e.,  $a \in \bar{A}$ .

Conversely, suppose  $a \in \bar{A}$ . Consider the sequence of balls  $O(a, 1/n)$ . In each of them there are points of  $A$  (part 1 of Proposition 1). Taking one such point  $a_n$  for each  $n$ , we obtain a sequence  $\{a_n\}$ , where  $a_n \in O(a, 1/n)$ . This sequence converges to  $a$ , since  $\rho(a, a_n) \rightarrow 0$  as  $n \rightarrow \infty$ . ■

DEFINITION 12. A mapping  $g$  of one metric space  $R = (X, \rho)$  into another  $R_0 = (Y, \rho_0)$  is said to be *continuous at the point  $x$*  if for each neighborhood  $\Sigma_{g(x)}$  of the point  $g(x)$  there is a neighborhood  $\Sigma_x$  of the

point  $x$  such that  $g(\Sigma_x) \subset \Sigma_{g(x)}$ . If  $g$  is continuous at each point, it is said to be *continuous on  $R$* .

LEMMA 4. A mapping  $g : R \rightarrow R_0$  is continuous if and only if the complete preimage of any open set is open.

PROOF: Let  $g$  be a continuous mapping and  $G_0$  an open set in  $R_0$ . If the complete preimage of  $G_0$  is not empty, then it is an open set. Indeed if  $a \in g^{-1}(G_0)$ , then, since  $g$  is continuous at the point  $a$ , there exists a neighborhood  $\Sigma_a$  of the point  $a$  such that  $g(\Sigma_a) \subset G_0$ , i.e.,  $\Sigma_a$  is contained in  $g^{-1}(G_0)$ , the complete preimage of  $G_0$ . Since  $g^{-1}(G_0) = \bigcup_{a \in g^{-1}(G_0)} \Sigma_a$ ,

it follows that  $g^{-1}(G_0)$  is an open set, being the union of open sets. If the complete preimage of  $G_0$  is empty, it is obviously open. Conversely, if the complete preimage of any open set in  $R_0$  is open, then, taking a point  $a \in X$  and an arbitrary neighborhood  $\Sigma_{g(a)}$  of its image, we find that  $g^{-1}(\Sigma_{g(a)})$  is an open set in  $R$  whose image is contained in  $\Sigma_{g(a)}$ , i.e.,  $g$  is continuous. ■

PROPOSITION 3. Let  $g : X \rightarrow Y$  be a mapping of the metric space  $X$  into the metric space  $Y$ . The mapping  $g$  is continuous if and only if it possesses the following property: If  $x_0, x_n \in X$ ,  $n = 1, 2, 3, \dots$ , and  $x_n \rightarrow x_0$ , then  $g(x_n) \rightarrow g(x_0)$ .

PROOF: Suppose  $g$  is continuous and  $x_n \rightarrow x_0$ . For each neighborhood  $\Sigma_{g(x_0)} \subset Y$  there exists a neighborhood  $\Sigma_{x_0} \subset X$  such that  $g(\Sigma_{x_0}) \subset \Sigma_{g(x_0)}$ . The neighborhood  $\Sigma_{x_0}$  contains all points  $x_n$  from some index  $n_0$  on:  $x_n \in \Sigma_{x_0}$  for  $n \geq n_0$ . But then  $g(x_n) \in g(\Sigma_{x_0}) \subset \Sigma_{g(x_0)}$  for  $n \geq n_0$ . Thus any neighborhood  $\Sigma_{g(x_0)}$  of the point  $g(x_0)$  contains all but a finite number of the terms  $\{g(x_n)\}$ , i.e.,  $g(x_n) \rightarrow g(x_0)$ .

Conversely let  $\Sigma \subset Y$  be an open set and  $G = \{x \in X : g(x) \in \Sigma\}$  its preimage. If  $G$  were not open, one of its points  $x_0$  would belong to the closure of the complement of  $G$ . Then (cf. Lemma 3) there would exist a sequence  $\{x_n\}$  converging to  $x_0$  with  $x_n \notin G$  for all  $n$ . Then we would have, on the one hand  $g(x_n) \notin g(G)$  and so  $g(x_n) \notin \Sigma$ , while on the other hand  $g(x_n) \rightarrow g(x_0) \in \Sigma$ . This contradicts the fact that  $\Sigma$  is open. ■

DEFINITION 13. A mapping  $g$  of a metric space  $X$  into a metric space  $Y$  is called a *homeomorphism* if  $g$  maps  $X$  onto  $Y$  in a one-to-one manner and  $g$  is continuous together with  $g^{-1}$ .

#### EXAMPLES

1. It is obvious that the mapping  $g : X \rightarrow X$  of a metric space  $X$  into itself defined by the rule  $g(x) = x$  for any  $x \in X$  is continuous. Such a mapping is called the identity mapping and is denoted by the symbol  $E$ .



2. The distance function  $\rho(x, y)$  mapping  $X \times X$  into  $\mathbf{R}^1$  is a continuous function. The fact that it is continuous follows from the rectangle inequality  $|\rho(x, z) - \rho(y, u)| \leq |\rho(x, y) + \rho(z, u)|$ , which is obtained in an obvious manner from the two inequalities  $\rho(x, z) \leq \rho(x, y) + \rho(y, z) \leq \rho(x, y) + \rho(y, u) + \rho(u, z)$  and  $\rho(y, u) \leq \rho(y, x) + \rho(x, u) \leq \rho(y, x) + \rho(x, z) + \rho(z, u)$  by subtracting  $\rho(y, u)$  from the first and  $\rho(x, z)$  from the second. For  $z = u$  we obtain a second triangle inequality

$$|\rho(x, z) - \rho(y, z)| \leq \rho(x, y).$$

## 2.5. Compactness

A cover of the set  $A$  in a metric space is defined to be any family of open sets whose union contains  $A$ .

DEFINITION 14. The metric space  $(X, \rho)$  [resp. a subset of a metric space] is called *compact* if every cover of it contains a finite subcover. The space  $R$  is called *locally compact* if each point of it has a neighborhood whose closure is compact.

An example of a compact metric space is the interval  $[0, 1]$ , regarded as a metric space with the usual Euclidean distance. An example of a locally-compact space is the space  $\mathbf{R}^1$  (or  $\mathbf{R}^n$ ,  $n > 1$ ) or, for example, the space  $\mathbf{C}$ —the complex plane with the usual distance.

DEFINITION 15. A system of subsets  $\{A_\alpha\}$  of the set  $A$  is called *centered* if any finite subfamily of the system has a nonempty intersection.

The following lemmas hold.

LEMMA 5. A necessary and sufficient condition for a metric space  $R = (X, \rho)$  to be compact is that each centered system of closed subsets of it have a nonempty intersection.

PROOF: Let the space  $X$  be compact and let  $\{F_\alpha\}$  be a centered system of closed subsets. The sets  $G_\alpha = X \setminus F_\alpha$  are open and no finite system of these sets  $G_{\alpha_n}$ ,  $1 \leq n \leq N < \infty$ , covers  $X$ . Then, since  $X$  is compact,  $\{G_\alpha\}$  cannot be a cover of the compact space  $X$ . Otherwise we would be able to choose a finite subcover  $\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$  of the space  $X$  from the system  $\{G_\alpha\}$ , and this would mean that  $F_{\alpha_1} \cap \dots \cap F_{\alpha_n} = \emptyset$ . But if  $\{G_\alpha\}$  does not cover  $X$ , then  $\bigcap_{\alpha} F_\alpha$  is not empty.

Conversely suppose any centered system of closed subsets of  $R$  has nonempty intersection. Let  $\{G_\alpha\}$  be an open cover of  $X$ . We set  $F_\alpha = X \setminus G_\alpha$  and remark that since  $\{G_\alpha\}$  covers all of  $X$ , we have  $\bigcap_{\alpha} F_\alpha = \emptyset$ . Thus  $\{F_\alpha\}$  is



not centered, i.e., there exist  $F_1, F_2, \dots, F_M$  such that  $\bigcap_{i=1}^M F_i = \emptyset$ ,  $M < \infty$ . But then  $\{G_i\}_{i=1}^M = \{X \setminus F_i\}_{i=1}^M$  is a finite subcover of the cover  $\{G_\alpha\}$ . ■

LEMMA 6. *A closed subset of a compact metric space is compact\*.*

PROOF: Let  $F$  be a closed subset of a compact metric space  $X$  and  $\{\Sigma_\alpha\}$  (some system of open sets) a cover of  $F$ . We adjoin to the system  $\{\Sigma_\alpha\}$  the open set  $G = X \setminus F$  and denote the cover of the entire space  $X$  so obtained by  $\{\Sigma_\alpha^1\} = \{\Sigma_\alpha\} \cup G$ . By virtue of the compactness of  $X$  we choose a finite cover of the whole space from the system  $\{\Sigma_\alpha^1\}$ —a system  $\{\Sigma_i^1\}_{i=1}^N$ . Eliminating the set  $G$  from the system  $\{\Sigma_i^1\}_{i=1}^N$  if necessary, we obtain a finite cover of the set  $F$  chosen from the system  $\{\Sigma_\alpha\}$ . ■

LEMMA 7. *The image of a compact space  $X$  under a continuous mapping is a compact space.*

PROOF: Let  $g$  be a continuous mapping of  $X$  onto  $Y$ . Let  $\{\Sigma_\alpha\}$  be a cover of  $Y$  by open sets and  $\Psi_\alpha = g^{-1}(\Sigma_\alpha)$ . The sets  $\Psi_\alpha$  are open (cf. Lemma 4) and  $\{\Psi_\alpha\}$  is a cover of  $X$ . By the compactness of  $X$  we choose a finite subcover of this cover:  $\{\Psi_i\}_{i=1}^M$ . Then  $\{\Sigma_i\}_{i=1}^M$ ,  $M < \infty$ , is a cover of  $Y$ , where  $\Sigma_i = g(\Psi_i)$ ,  $i = 1, \dots, M$ . ■

LEMMA 8. *A compact subset, regarded as a subspace of a metric space  $X$ , is closed.*

PROOF: Let  $F$  be a compact subset and let  $a \in X \setminus F$ . For any point  $x \in F$  there exist neighborhoods  $\Sigma_a$  and  $\Sigma_x$  of the points  $a$  and  $x$  respectively such that  $\Sigma_a \cap \Sigma_x = \emptyset$ . One can take as such neighborhoods, for example, the balls  $O(a, r)$  and  $O(x, r)$ ,  $r = \frac{1}{3}\rho(a, x)$ . The set  $G = \bigcup_{x \in F} \Sigma_x$  is a cover of the set  $F$ . Since  $F$  is compact, we choose a finite subcover of this cover:  $\{\Sigma_{x_i}\}_{i=1}^M$ . Consider the neighborhoods  $\Sigma_a^i$  corresponding to  $\Sigma_{x_i}$ , which, by construction, do not intersect  $\Sigma_{x_i}$  and are such that  $\bigcap_{i=1}^M \Sigma_a^i = \Sigma$  is a neighborhood of the point  $a$ . It is obvious that  $\Sigma \cap \Sigma_{x_i} = \emptyset$ ,  $i = 1, \dots, M$  and therefore  $\Sigma \cap F = \emptyset$ . Thus  $\Sigma \subset X \setminus F$ , i.e., the set  $X \setminus F$  is open and  $F$  is closed. ■

LEMMA 9. *Let  $g : X \rightarrow \mathbf{R}^1$ , where  $\mathbf{R}^1$  is the real line. If  $g$  is a continuous mapping and  $X$  is compact, then  $g$  is bounded and attains its least upper bound and its greatest lower bound.*

\*A set whose closure is compact is called *precompact*.

PROOF: Let  $g(X)$  be the continuous image of a compact set (or space). By Lemma 7, the subset  $g(X)$  of the metric space  $\mathbf{R}^1$  is compact, hence closed and bounded. This obviously means that there exists a nonnegative number  $T$  such that  $|g(x)| \leq T$  and  $g(X)$  contains its least upper bound and its greatest lower bound. ■

We have given above (cf. Definition 8) the concept of a relatively open and a relatively closed set. We emphasize once again that the concepts of openness or closedness of a set are relative in the sense that a given set may be open in one space and not open in another space containing the first one.

We give another example illustrating this. The segment  $I = [0, 1]$  is an open set in the metric space  $(Y, \rho)$ , where  $Y = [0, 1]$  and  $\rho$  is the usual Euclidean metric on the segment. On the other hand this same segment  $[0, 1]$  is not open in the metric space  $(X, \rho)$ , where  $X = (-\infty, \infty)$ , and  $\rho$  is the same function. We remark that  $Y$  is a subspace of the space  $X$ ,  $I = Y \subset X$ .

However, the concept of a compact set is an absolute concept, i.e., independent of the ambient space. We have

PROPOSITION 4. *Let  $(F, \rho)$ ,  $(Y, \rho)$ ,  $(X, \rho)$  be metric spaces with  $F \subset Y \subset X$ . If the set  $F$  is compact in one of the spaces  $Y$  and  $X$ , it is compact in the other.*

PROOF: Assume that  $F$  is compact in  $X$ . Let  $\{\Sigma_\alpha\}$  be a family of subsets that are open relative to  $Y$ , and let  $F \subset \bigcup_\alpha \Sigma_\alpha$ . According to Definition 8 for each  $\alpha$  there exists a set  $G_\alpha$  open relative to  $X$  such that  $\Sigma_\alpha = Y \cap G_\alpha$ . Since  $F$  is compact in  $X$ , we have  $F \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$  for some choice of a finite number of indices  $\alpha_1, \dots, \alpha_n$ . Since  $F \subset Y$ , it follows from the last inclusion that  $F \subset \Sigma_{\alpha_1} \cup \cdots \cup \Sigma_{\alpha_n}$ . It is thereby proved that the set  $F$  is compact in  $(Y, \rho)$ .

Now let  $F$  be compact in  $Y$ . Let  $\{G_\alpha\}$  be an open cover of  $F$  in  $X$ , and let  $\Sigma_\alpha = Y \cap G_\alpha$ . Then there exists a finite number of indices  $\alpha_1, \dots, \alpha_n$  such that  $F \subset \Sigma_{\alpha_1} \cup \cdots \cup \Sigma_{\alpha_n}$ . Since  $\Sigma_\alpha \subset G_\alpha$ , we have the inclusion  $F \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$ . Hence  $F$  is compact in  $X$ . ■

Setting  $Y = F$  in this assertion, we obtain the result that  $F$  is compact relative to any ambient space if and only if it is compact "relative to itself," i.e., simply compact.

## 2.6. A Basis for the Topology of a Space

DEFINITION 16. A system of open sets  $\{\Sigma_\alpha\}$  in a metric space  $(X, \rho)$  is called a *basis for the topology* of the space if every nonempty open set of the space  $X$  can be obtained as the union of sets in the system  $\{\Sigma_\alpha\}$ .



The simplest example of a basis for a topology (or a *basis* for short) is the set of all open sets of the given space. The following fact holds, which makes it possible to establish whether a given system is a basis of the space.

LEMMA 10. *A necessary and sufficient condition for a system  $\{\Sigma_\alpha\}$  to be a basis of the space  $(X, \rho)$  is that for every open set  $G$  and every point  $a \in G$  there exist a set  $\Sigma_{\alpha_0}$  of the given system such that  $a \in \Sigma_{\alpha_0} \subset G$ .*

PROOF: Let  $\{\Sigma_\alpha\}$  be a basis of the space  $X$  and let  $G$  be an arbitrary open set and  $a \in G$ . Then there exists a subsystem  $\{\Sigma_{\alpha_k}\}$  such that  $G = \bigcup_k \Sigma_{\alpha_k}$ . Hence there exists  $\Sigma_{\alpha_0}$  such that  $a \in \Sigma_{\alpha_0} \subset \bigcup_k \Sigma_{\alpha_k} = G$ .

Conversely if the hypotheses of the lemma are satisfied, then for any point  $x \in G$  there exists a neighborhood  $\Sigma_x$  of the system  $\{\Sigma_\alpha\}$  such that  $x \in \Sigma_x \subset G$ . Then  $G = \bigcup_{x \in G} \Sigma_x$  and therefore  $\{\Sigma_\alpha\}$  is a basis of the space  $X$ . ■

Thus in a metric space the set of open balls, for example, forms a basis.

DEFINITION 17. A metric space  $(X, \rho)$  is called *second-countable* if there exists a basis of it consisting of at most a countable number of sets. Second-countable spaces are said to satisfy the *second axiom of countability*.

LEMMA 11. *A metric space  $(X, \rho)$  is second-countable if it contains a countable everywhere-dense set. Conversely, if the space  $X$  has a countable basis, then it contains a countable everywhere-dense set.*

PROOF: Let  $A = \{a_n\}_1^\infty$  be a countable everywhere-dense set in  $X$ . The set of open balls  $O(a_n, 1/m)$ , where  $n$  and  $m$  range over the natural numbers, forms a basis of the space which is countable.

Conversely, if  $X$  contains a countable basis  $\{\Sigma_n\}_{n=1}^\infty$ , then, choosing one point from each set ( $a_n \in \Sigma_n$ ), we obtain a set  $A = \{a_n\}_{n=1}^\infty$  that is everywhere-dense. Indeed, if  $\bar{A} \neq X$ , then the open set  $G = X \setminus \bar{A}$  would be nonempty and would not contain any point of  $A = \{a_n\}$ , which is impossible, since  $G$  is an open set and is therefore the union of some sets in the system  $\{\Sigma_n\}$ , while  $a_n \in \Sigma_n$ . ■

#### EXAMPLES

1. One can construct a metric space  $(X, \rho)$  and closed balls  $K_1(x_1, r_1)$  and  $K_2(x_2, r_2)$  such that  $K_1 \subset K_2$ , and  $r_1 > r_2$ .

Indeed, let  $(X, \rho)$  be a metric space consisting of the points  $(x, y)$  of a closed disk in the  $xy$ -plane:  $X \equiv \{(x, y) : x^2 + y^2 \leq 9\}$  with the usual Euclidean metric. We define the ball  $K_2$  as follows:  $K_2 \equiv (X, \rho)$ . Let the ball  $K_1 \equiv K_2 \cap \{(x, y) : (x-2)^2 + y^2 \leq 16\}$ . then  $K_1 \subset K_2$ ,  $r_1 = 4$ ,  $r_2 = 3$ ,  $r_1 > r_2$ .



2. A set  $A$  in a metric space is closed if and only if it coincides with its closure  $\bar{A}$ , i.e.,  $a = \bar{A}$ .

Indeed, if  $A = \bar{A}$ , then, since the closure of any set is closed (as an intersection of closed sets),  $A$  is closed. Conversely  $A \subset \bar{A}$  and if  $A$  is closed, then it contains the intersection of all closed sets containing  $A$ , i.e.,  $\bar{A} \subset A$ . On the other hand, by the definition of closure  $A \subset \bar{A}$ , i.e., if  $A$  is closed, then  $\bar{A} = A$ .

It has thereby been shown that a set is closed if and only if it contains all of its limit points (cf. Proposition 2).

3. A ball  $O(x_0, r)$  in a metric space  $X$  is an open set.

If  $x \in O(x_0, r)$ , i.e.,  $\rho(x, x_0) < r$ , then the ball  $O(x, \varepsilon)$  will be contained in the initial set for  $0 < \varepsilon < r - \rho(x_0, x)$ :  $O(x, \varepsilon) \subset O(x_0, r)$ . Indeed, if  $y \in O(x, \varepsilon)$ , i.e.,  $\rho(x, y) < \varepsilon$ , then

$$\rho(x_0, y) \leq \rho(x_0, x) + \rho(x, y) < \rho(x_0, x) + r - \rho(x_0, x) = r.$$

In exactly the same way a closed ball—a set  $K(x, r) = \{y : \rho(x, y) \leq r\}$ —is a closed set in  $X$ . This follows from the fact that its complement is an open set.

4. In a metric space  $(X, \rho)$  one can construct an open ball  $O(x, r) = \{y : \rho(x, y) < r\}$  and a closed ball  $K(x, r) = \{y : \rho(x, y) \leq r\}$  with a common center and equal radii such that

$$\bar{O}(x, r) \neq K(x, r).$$

Indeed, let  $X$  be a set consisting of more than one point and let

$$\rho(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Consider the metric space  $(X, \rho)$ . Let  $x$  be an arbitrary point of  $X$ . Then  $O(x, 1) = \{x\}$  and  $K(x, 1) = X$ . Since here the ball  $O(x, 1)$  has no limit points, we have  $\bar{O}(x, 1) = O(x, 1) \neq K(x, 1)$ .

5. Let  $\mathbf{R}^2$  be a two-dimensional plane with the usual Euclidean distance. Let  $O_1$ —a subspace of  $\mathbf{R}^2$ —be the unit circle:  $x^2 + y^2 = 1$ . We denote by  $I_1$  the interval  $[0, 2\pi)$  of the real line. We define a mapping of the set  $I = [0, 2\pi)$  onto the unit circle  $O_1$  using the formula  $x = \cos \varphi$ ,  $y = \sin \varphi$ ,  $\varphi \in I_1$ . It is obvious that this mapping is continuous and one-to-one, but the inverse mapping of the space  $O_1$  onto the space  $I_1$  is not continuous at the point with coordinates  $x = 1$ ,  $y = 0$ .

6. If a sequence  $\{a_n\}$  of points of a set  $Y$  in a metric space converges to the point  $a$  in the metric space, then the point  $a$  is either a limit point for the set  $Y$  or an isolated point of the set  $Y$ .

Indeed if  $a_n \rightarrow A$ , then any neighborhood  $\Sigma_a$  of the point  $a$  contains all but a finite number of points of the sequence  $\{a_n\}$ . Therefore either any neighborhood  $\Sigma_a$  of the point  $a$  contains a point of the sequence  $a_n \in Y$ ,  $a_n \neq a$ , and then  $a$  is a limit point of  $Y$ , or there exists a neighborhood of the point  $a$  in which there are no points of the sequence  $\{a_n\}$  different from  $a$ , and then  $a$  is an isolated point of the set  $Y$ .

7. Let  $(X, \rho)$  and  $(Y, \rho_1)$  be metric spaces and  $g : X \rightarrow Y$ . The mapping  $g$  is continuous at the point  $a \in X$  if and only if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\rho_1(g(x), g(a)) < \varepsilon$  for any  $x$  such that  $\rho(x, a) < \delta$ .

Indeed, fix  $a \in X$  and  $\varepsilon > 0$  and let  $O_0(g(a), \varepsilon)$  be a ball in the space  $Y$  of radius  $\varepsilon$  and with center at  $g(a) : \{y : \rho(g(a), y) < \varepsilon\}$ .  $O_0(g(a), \varepsilon)$  is an open set in  $Y$ . Therefore there exists  $\delta > 0$  such that the point  $a$  belongs to  $g^{-1}(O_0)$  together with the ball  $O(a, \delta)$  of radius  $\delta : \{x : \rho(a, x) < \delta\}$ . But if  $x \in g^{-1}(O_0)$ , then  $g(x) \in O_0$  and therefore  $\rho_1(g(a), g(x)) < \varepsilon$ . The converse is obvious. We remark that if  $a$  is an isolated point of the set  $A \subset X$ , then it follows from what has been proved that any mapping  $g$  defined at this point is continuous there: for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that in the ball  $\rho(a, x) < \delta$  there will be only the one point  $x = a$ ; then  $\rho_1(g(x), g(a)) = 0 < \varepsilon$ .

8. A metric space is compact if and only if for any centered system  $\{\Sigma_\alpha\}$  of subsets  $\bigcap_{\alpha} \overline{\Sigma}_\alpha \neq \emptyset$ .

In fact, if the hypotheses of this proposition hold and  $\{F_\alpha\}$  is an arbitrary centered system of closed subsets of the space, then according to Example 2 given above and Lemma 5, the space is compact.

Conversely suppose the space is compact and let  $\{\Sigma_\alpha\}$  be an arbitrary centered system of subsets of it. Consider the system  $\{\overline{\Sigma}_\alpha\}$ , where  $\Sigma_\alpha \subset \overline{\Sigma}_\alpha$ . The system  $\overline{\Sigma}_\alpha$  is centered and according to Lemma 5  $\bigcap_{\alpha} \overline{\Sigma}_\alpha \neq \emptyset$ , since the sets  $\overline{\Sigma}_\alpha$  are closed (cf. Example 2).

## EXERCISES

1. Let  $A$  and  $B$  be two subsets of a metric space  $X$ . The number  $\rho(A, B) = \inf_{a \in A, b \in B} \rho(a, b)$  is called the *distance between the subsets  $A$  and  $B$*  in  $X$ . Give an example of two subsets  $A$  and  $B$  in a metric space  $X$  such that  $A \cap B = \emptyset$  but  $\rho(A, B) = 0$ .

2. Prove that the set  $A$  of continuous functions  $f(x)$  on the interval  $[0, 1]$



satisfying the inequalities  $a < f(x) < b$ , where  $a < b$  are given numbers, is an open set in  $C[0, 1]$ .

3. Consider the metric space  $(X, \rho)$ , where  $X$  is the real line and  $\rho$  the usual Euclidean metric (the space  $\mathbf{R}^1$ ). Let  $A = \{2^{p/q}\}$ , where  $p$  and  $q$  range over all natural numbers. Find the closure of the set  $A$ .

4. Prove that the set of points of the form  $\sin r$ , where  $r$  ranges over the rational numbers in the interval  $[-\pi/2, \pi/2]$  is everywhere-dense in the interval  $[-1, 1]$ .

5. Show that in the space  $C[a, b]$  there exist closed bounded sets that are not compact in  $C[a, b]$ . (A set  $A$  in a metric space  $(X, \rho)$  is *bounded* if there exists a number  $N > 0$  such that  $\rho(x, a) \leq N$  for any point  $a \in A$ , where  $x$  is some point of the space  $X$ .)

6. Construct a nonempty perfect set on the line  $\mathbf{R}^1$  all of whose points are irrational.

7. Prove that on the line  $\mathbf{R}^1$  the only connected sets are intervals (including infinite intervals) that are open, half-open, or closed.

8. A mapping  $g$  of one metric space  $(X, \rho)$  onto another  $(Y, \rho_0)$  is called *open* if any open set  $A$  in  $X$  maps to an open set  $g(A)$  in  $Y$ . Prove that a mapping  $g$  is open if and only if for each point  $a \in X$  and any neighborhood  $\Sigma_a$  of it in  $X$  there exists a neighborhood  $\Sigma_{g(a)}$  such that  $\Sigma_{g(a)} \subset g(\Sigma_a)$ .

9. Let  $(X, \rho)$  be a metric space and  $(Y, \rho)$  a subspace of it. Let the system  $\{\Sigma_\alpha\}$  be a basis in  $X$ . Denote by  $\{\Sigma_\alpha^0\}$  the collection of all sets of the form  $\Sigma_\alpha \cap Y$ . Then  $\{\Sigma_\alpha^0\}$  is a basis in  $Y$ . Prove this.

10. In the set  $X$  of continuous functions defined on the closed interval  $[a, b]$  define two distance functions  $\rho$  and  $\rho_0$  such that the complement of the unit ball in the space  $(X, \rho)$  is everywhere dense in the unit ball of the space  $(X, \rho_0)$ .

### 3. PROPERTIES OF METRIC SPACES

In the preceding section we discussed the basic properties of metric spaces, basing the exposition on the concept of open and closed sets. It should be emphasized that essentially all the propositions of the preceding section use only the properties of open sets: the union of any number of open sets and the intersection of any finite number of open sets is open, and the whole space and the empty set are open sets. These propositions do not use such concepts as ball or distance. At the same time, since the



distance function is introduced into the structure of a metric space, these spaces must possess certain properties peculiar to them. Moreover, it is most often precisely these properties that are studied when metric spaces are considered.

In the present section we discuss the fundamental properties of metric spaces. They all use the concept of completeness of a space.

**DEFINITION 1.** A sequence  $\{x_n\}$ ,  $n = 1, 2, \dots$  of elements of a metric space  $(X, \rho)$  is called *fundamental*\* if  $\rho(x_n, x_m) \rightarrow 0$  when  $n, m \rightarrow \infty$ ,  $n$  and  $m$  being natural numbers.

We remark that, as already stated, if the sequence  $\{x_n\}$  converges to the element  $x$  of the space, then  $\rho(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ .

It is also obvious that if the sequence  $\{x_n\}$  converges to the element  $x$ , then by the triangle inequality  $\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x_m, x) \rightarrow 0$  as  $n, m \rightarrow \infty$ , i.e., the sequence  $\{x_n\}$  is fundamental.

On the other hand, not every fundamental sequence  $\{x_n\}$  of elements of a metric space  $(X, \rho)$  is a convergent sequence in the given space.

Indeed, consider for example as the metric space  $(X, \rho)$  the interval  $(0, 1) = X$  with the usual distance  $\rho$  between numbers of the interval. The sequence  $\{1/n\}$ ,  $n = 1, 2, \dots$ , is obviously fundamental, since  $\rho\left(\frac{1}{n}, \frac{1}{m}\right) = \left|\frac{1}{n} - \frac{1}{m}\right| \rightarrow 0$  as  $n, m \rightarrow \infty$ . But this sequence does not converge to any element of the set  $X = (0, 1)$ , i.e., is not convergent in the space  $(X, \rho)$ .

In connection with this we give the following definition.

**DEFINITION 2.** A metric space  $(X, \rho)$  is called *complete* if every fundamental sequence in it converges to some limit, which is an element of the space.

The example just given shows that not every metric space is complete. Therefore the question arises: Is there a way of completing an incomplete metric space?

An affirmative answer to this question will be given below.

We now give examples of complete metric spaces.

#### EXAMPLES

1. The completeness of the space  $\mathbf{R}^n$  is a consequence of the completeness of  $\mathbf{R}^1$  (the real numbers). Indeed, let  $\{\xi^{(p)}\}$  be a fundamental sequence

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\*Fundamental sequences are more frequently called *Cauchy* sequences in the West. Tr.

of elements of  $\mathbf{R}^n$ . Then for any  $\varepsilon > 0$  there exists  $N = N(\varepsilon)$  such that

$$\rho(\xi^{(p)}, \xi^{(q)}) = \left( \sum_{k=1}^n (\xi_k^{(p)} - \xi_k^{(q)})^2 \right)^{1/2} < \varepsilon$$

for all  $p, q > N$ . Here

$$\xi^{(p)} = (\xi_1^{(p)}, \xi_2^{(p)}, \dots, \xi_n^{(p)}), \quad \xi^{(q)} = (\xi_1^{(q)}, \xi_2^{(q)}, \dots, \xi_n^{(q)}).$$

Then *a fortiori* for any  $k = 1, 2, \dots$

$$|\xi_k^{(p)} - \xi_k^{(q)}| < \varepsilon,$$

i.e.,  $\{\xi_k^{(p)}\}$  is a fundamental numerical sequence. Let  $\xi_k = \lim_{p \rightarrow \infty} \xi_k^{(p)}$  and  $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ . It is obvious that  $\lim_{p \rightarrow \infty} \xi^{(p)} = \xi$ .

2. We shall establish that the space  $C[a, b]$  is complete. Let  $\{x_n\}$  be a fundamental sequence in  $C[a, b]$ . Then for any  $\varepsilon > 0$  there exists  $N$  such that

$$\max_{a \leq t \leq b} |x_n(t) - x_m(t)| < \varepsilon$$

It follows from this that the sequence  $\{x_n(t)\}$  converges uniformly on  $[a, b]$ . In this case its limit  $x(t)$  is a continuous function. Letting  $m$  tend to infinity in the inequality  $|x_n(t) - x_m(t)| < \varepsilon$ , we obtain

$$|x_n(t) - x(t)| \leq \varepsilon$$

for all  $t \in [a, b]$  and for  $n > N$ . Consequently  $\{x_n(t)\}$  converges to  $x(t)$  in the sense of the metric of the space  $C[a, b]$ .

### 3.1. Completion of Metric Spaces

**DEFINITION 3.** A one-to-one mapping  $g$  of one metric space  $(X, \rho)$  onto another  $(Y, \rho_0)$  is called an *isometry* if for any points  $x_1, x_2 \in X$  the relation  $\rho(x_1, x_2) = \rho_0(g(x_1), g(x_2))$  holds. In this case the spaces  $(X, \rho)$  and  $(Y, \rho_0)$  are said to be *isometric* to each other.

**DEFINITION 4.** A complete metric space  $(Y, \rho_0)$  is called the *completion* of the metric space  $(X, \rho)$  if  $(X, \rho)$  is a subspace of  $(Y, \rho_0)$  and the closure of the subspace  $(X, \rho)$  coincides with all of  $(Y, \rho_0)$ .

The following theorem holds.



**THEOREM 1.** *For any metric space  $(X, \rho)$  there exists a completion  $(Y, \rho_0)$  of it. This completion  $(Y, \rho_0)$  is unique up to an isometry.*

**PROOF:** Let  $(X, \rho)$  be an arbitrary metric space. We call two fundamental sequences  $\{x_n\}$  and  $\{x'_n\}$  of  $X$  equivalent and denote this fact  $\{x_n\} \sim \{x'_n\}$  if  $\lim_{n \rightarrow \infty} \rho(x_n, x'_n) = 0$ . This notion of equivalence is reflexive, symmetric, and transitive. Hence the fundamental sequences that can be formed from points of the space  $X$  break up into classes of mutually equivalent sequences. We now define the space  $(Y, \rho_0)$ . We take as  $Y$  the set of all equivalence classes of sequences. We denote these classes as  $x_0, y_0, \dots$ , and introduce a notion of distance by the rule  $\rho(x_0, y_0) = \lim_{n \rightarrow \infty} \rho(x_n, y_n)$ , where  $\{x_n\}$  and  $\{y_n\}$  are arbitrary fundamental sequences from the classes  $x_0$  and  $y_0$  respectively. The indicated limit exists and is independent of the choice of sequences  $\{x_n\}$  and  $\{y_n\}$ . In fact  $|\rho(x_n, y_n) - \rho(x_m, y_m)| \leq \rho(y_n, y_m) + \rho(x_n, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ , if  $n, m \geq N(\varepsilon)$ , since the sequences are fundamental. Therefore the numerical sequence  $\rho(x_n, y_n)$  is fundamental and has a limit since  $\mathbf{R}^n$  is complete. Now let  $\{x_n\}$  and  $\{x'_n\} \in x_0$  and  $\{y_n\}$  and  $\{y'_n\} \in y_0$ . Then again  $|\rho(x_n, y_n) - \rho(x'_n, y'_n)| \leq \rho(x_n, x'_n) + \rho(y_n, y'_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ , if  $n \geq N(\varepsilon)$ , since  $\{x_n\} \sim \{x'_n\}$  and  $\{y_n\} \sim \{y'_n\}$ . Thus  $\lim_{n \rightarrow \infty} \rho(x_n, y_n) = \lim_{n \rightarrow \infty} \rho(x'_n, y'_n)$  and the number  $\rho_0(x_0, y_0)$  is independent of the choice of the representatives of the classes  $x_0$  and  $y_0$ .

Obviously all the axioms of a metric space hold in the space  $Y$ . Thus the triangle inequality is obtained from the triangle inequality in  $X$  by passing to the limit.

We shall now show that the metric space  $Y$  is the completion of the space  $X$ . Let  $x \in X$ . Consider the constant sequence  $\{x_n\}$ ,  $x_n = x$ . It is obviously a fundamental sequence. Denote by  $\tilde{x}$  the element of  $Y$  that is the class of fundamental sequences equivalent to it. It is immediately verified that for any  $x_1$  and  $x_2 \in X$  the relation  $\rho(x_1, x_2) = \rho_0(\tilde{x}_1, \tilde{x}_2)$  holds. This means that the space  $X$  is isometric to its image  $\tilde{X} \subset Y$  under the mapping  $x \rightarrow \tilde{x}$ . By identifying  $X$  and  $\tilde{X}$  we may assume that  $X$  is a subspace of  $Y$ . We shall now prove that it is dense in  $Y$ . Let  $x_0 \in Y$  and  $\varepsilon > 0$ . Choose some fundamental sequence  $\{x_n\}$  from the class  $x_0$ . Then there exists  $N$  such that  $\rho(x_n, x_m) < \varepsilon$  for any  $n, m > N$ . Take an element  $\tilde{x}_N = \{x_N, x_N, \dots, x_N, \dots\} \in \tilde{X}$ . Then  $\rho_0(x_0, \tilde{x}_N) < \varepsilon$ .

We shall now prove that  $Y$  is complete. Let  $x_0^1, x_0^2, \dots$  be a fundamental sequence of points in  $Y$ . Since  $X$  is dense in  $Y$ , we can find points  $x_n \in X$  such that  $\rho_0(\tilde{x}_n, x_0^n) < \frac{1}{n}$ . From the triangle inequality we obtain

$$\begin{aligned}\rho(x_m, x_n) &= \rho_0(\tilde{x}_m, \tilde{x}_n) \leq \rho_0(\tilde{x}_m, x_0^m) + \rho_0(\tilde{x}_n, x_0^n) + \rho_0(x_0^m, x_0^n) \\ &\leq \frac{1}{m} + \frac{1}{n} + \rho_0(x_0^m, x_0^n).\end{aligned}$$

It follows from this that  $\{x_n\}$  is fundamental in  $X$ . Hence a certain element of  $Y$  corresponds to it (by definition of  $Y$ ). Denote this element by  $x_0$ . We further have:

$$\rho_0(x_0^n, x_0) \leq \rho_0(x_0^n, \tilde{x}_n) + \rho_0(\tilde{x}_n, x_0).$$

Each of the terms on the right-hand side tends to zero as  $n \rightarrow \infty$ , the first because of the choice of the points  $x_n$  and the second because  $\{x_n\}$  is a fundamental sequence in  $X$ . Hence  $x_0$  is the limit (in  $Y$ ) of the sequence  $\{x_0^n\}$ , and so  $Y$  is complete.

It remains only to prove that the completion is unique. Let  $(Y, \rho_0)$  and  $(Y', \rho'_0)$  be two completions of the space  $(X, \rho)$ . Let  $x_0$  be an arbitrary point of the space  $(Y, \rho_0)$ . Then there exists a sequence  $\{x_n\}$  of points of  $(X, \rho)$  converging to  $x_0$ . The points  $x_n$  belong to  $(Y', \rho'_0)$  also. Since  $Y'$  is complete, and the sequence  $\{x_n\}$  is fundamental,  $\{x_n\}$  converges also in  $Y'$  to some point  $x'_0$ , and this point is independent of the choice of the sequence  $\{x_n\}$  converging to  $x_0$ . We define a mapping  $g$  of the space  $Y$  onto  $Y'$  by the rule  $g(x_0) = x'_0$ . This is an isometric mapping of the space  $Y$  onto  $Y'$  such that  $g(x) = x$  for any  $x \in X$ . Indeed, let

$$\begin{aligned}\{x_n\} &\rightarrow x_0 \quad \text{in } Y, & \{x_n\} &\rightarrow x'_0 \quad \text{in } Y', \\ \{y_n\} &\rightarrow y_0 \quad \text{in } Y, & \{y_n\} &\rightarrow y'_0 \quad \text{in } Y' .\end{aligned}$$

Then

$$\rho(x_0, y_0) = \lim_{n \rightarrow \infty} \rho_0(x_n, y_n) = \lim_{n \rightarrow \infty} \rho(x_n, y_n)$$

and

$$\rho'_0(x'_0, y'_0) = \lim_{n \rightarrow \infty} \rho'_0(x_n, y_n) = \lim_{n \rightarrow \infty} \rho(x_n, y_n).$$

Therefore  $\rho_0(x_0, y_0) = \rho'_0(x'_0, y'_0)$  and by construction the mapping is one-to-one and onto, i.e., we have obtained an isometry. The equality  $g(x) = x$  for any  $x \in X$  is obvious from the construction. ■

### 3.2. Basic Theorems on Complete Metric Spaces

In this section we shall prove the basic theorems on complete metric spaces, such as the nested ball theorem, the Baire category theorem, and



the contraction mapping principle. These theorems are of great significance in the study of metric spaces and also will be the most frequently used in what follows.

The following theorem holds.

**THEOREM 2** (The nested ball theorem). *A necessary and sufficient condition for a metric space to be complete is that every nested sequence of closed balls whose radii tend to zero have a non-empty intersection.*

**PROOF.** **NECESSITY:** Let  $(X, \rho)$  be a complete metric space, and let  $K_1(x_1, r_1) \supset K_2(x_2, r_2) \supset \dots$  be a nested sequence of closed balls. The sequence of their centers is fundamental, since  $\rho(x_n, x_m) < r_n$  for  $m > n$  and  $r_n \rightarrow 0$  as  $n \rightarrow \infty$ . Since the space  $X$  is complete, there exists an element  $x = \lim_{n \rightarrow \infty} x_n$ ,  $x \in X$ . It is obvious that  $x \in \bigcap_{n=1}^{\infty} K_n$ . Indeed,  $x$  is a limit point of  $K_n$  for any  $n$ . Since all  $K_n$  are closed,  $x$  belongs to  $K_n$  for any  $n$ , i.e.,  $x \in \bigcap_{n=1}^{\infty} K_n$  (cf. Lemma 3 of Sec. 1.2.4).

**SUFFICIENCY:** We must show that if  $\{x_n\}$  is a fundamental sequence, it has a limit  $x \in X$ . Choose the point  $x_{n_1}$  such that  $\rho(x_n, x_{n_1}) < \frac{1}{2}$  for any  $n > n_1$ . Take  $x_{n_1}$  as the center of a closed ball of radius 1:  $K(x_{n_1}, 1)$ . We then choose the point  $x_{n_2}$  from the sequence  $\{x_n\}$  satisfying the following conditions:  $\rho(x_n, x_{n_2}) < \frac{1}{2^2}$  for any  $n > n_2$ ,  $n_2 > n_1$ . We choose the point  $x_{n_2}$  as the center of a ball of radius  $\frac{1}{2}$ :  $K(x_{n_2}, \frac{1}{2})$ . Let  $x_{n_1}, x_{n_2}, \dots, x_{n_k}$  ( $n_1 < n_2 < \dots < n_k$ ) be chosen. We then choose  $x_{n_{k+1}}$  so that the following conditions are satisfied:  $\rho(x_n, x_{n_{k+1}}) < \frac{1}{2^{k+1}}$  for any  $n \geq n_{k+1}$ ,  $n_{k+1} > n_k$ . As above, we take  $x_{n_{k+1}}$  as the center of a closed ball of radius  $\frac{1}{2^k}$ :  $K(x_{n_{k+1}}, \frac{1}{2^k})$ , etc. We have obtained a nested sequence of closed balls whose radii tend to zero. By hypothesis there exists a point  $x$  common to all the balls. It is clear that  $\rho(x_{n_k}, x) \rightarrow 0$  as  $n_k \rightarrow \infty$ . Thus the fundamental sequence  $\{x_n\}$  contains a subsequence  $\{x_{n_k}\}$  converging to some point  $x$  of the space. Then the original sequence converges to the same limit. Indeed, applying the triangle inequality for the distance function (cf. Sec. 1.2.1, Definition 1), we have

$$\rho(x_n, x) \leq \rho(x_n, x_{n_k}) + \rho(x_{n_k}, x) \rightarrow 0, \quad n, n_k \rightarrow \infty,$$

i.e., the space  $X$  is complete—every fundamental sequence converges to a limit that belongs to the space. ■

**REMARK.** All the hypotheses of the theorem are essential: the space must be complete, the balls closed and nested, and the radii must tend to

zero. This last condition is the least obvious. The following is an example of a complete metric space and a sequence of nested balls having an empty intersection.

Let  $R = (\mathbb{N}, \rho)$ , where  $\mathbb{N}$  is the set of natural numbers and

$$\rho(m, n) = \begin{cases} 1 + \frac{1}{n+m}, & \text{if } n \neq m, \\ 0, & \text{if } n = m. \end{cases}$$

We define a sequence of closed balls with centers at the points  $n$  and radius  $1 + \frac{1}{2n}$ :

$$K(n, 1 + \frac{1}{2n}) = \{m : \rho(m, n) \leq 1 + \frac{1}{2n}\}, \quad n = 1, 2, \dots$$

It is then obvious that the balls  $K(n, 1 + \frac{1}{2n})$  are closed and nested, and the space is complete, since each fundamental sequence converges in the space. (Such a sequence is "almost constant," in colloquial language.) However the intersection of these balls is empty.

**DEFINITION 5.** A subset  $M$  of a metric space  $X$  is called a *set of first category* if it can be represented as the union of a countable collection of sets that are nowhere dense in  $X$ .

All other sets are called *sets of second category*.

The following theorem holds.

**THEOREM (The Baire category theorem).** *Let  $(X, \rho)$  be a nonempty metric space. Then  $X$  is a set of second category, i.e.,  $X$  cannot be represented as the union of a countable collection of nowhere dense sets.*

**PROOF:** Suppose, to the contrary, that  $X = \bigcup_{n=1}^{\infty} A_n$  and each of the sets  $A_n$ ,  $n = 1, 2, \dots$  is nowhere dense in  $X$ . Let  $K_0$  be some closed ball of radius 1. Since the set  $A_1$  is nowhere dense, there exists a closed ball  $K_1$  of radius less than  $1/2$  such that  $K_1 \subset K_0$  and  $K_1 \cap A_1 = \emptyset$ . Since the set  $A_2$  is nowhere dense, in exactly the same way, there exists a closed ball  $K_2 \subset K_1$  of radius less than  $1/2^2$  for which  $K_2 \cap A_2 = \emptyset$ , etc. As a result we obtain a nested sequence of closed balls  $\{K_n\}_{n=1}^{\infty}$  whose radii tend to zero. By Theorem 1 there exists a point  $x \in X$  that belongs to  $K_n$  for any integer  $n$ . Since by construction  $K_n \cap A_n = \emptyset$ , it follows that for any  $n$  we have  $x \notin A_n$ . Therefore  $x \notin \bigcup_{n=1}^{\infty} A_n$ . This contradicts the assumption that

$$X = \bigcup_{n=1}^{\infty} A_n. \quad \blacksquare$$



**DEFINITION 6.** A mapping  $g$  of a metric space  $X$  into itself is called a *contraction* if there exists a number  $0 < \alpha < 1$  such that  $\rho(g(x), g(y)) < \alpha\rho(x, y)$  for any  $x, y \in X$ .

**THEOREM 4** (Contraction mapping principle). *Every contraction mapping of a complete metric space  $(X, \rho)$  into itself has one and only one fixed point, i.e., a point  $x \in X$ , such that  $g(x) = x$ .*

**PROOF:** Let  $x_0$  be some point of  $X$ . We define a sequence of points  $\{x_n\}_{n=1}^{\infty}$  according to the rule  $x_1 = g(x_0), \dots, x_n = g(x_{n-1})$ . The sequence  $\{x_n\}$  is fundamental in  $X$ . Indeed, if  $m > n$ , then

$$\begin{aligned} \rho(x_n, x_m) &= \rho(g(x_{n-1}), g(x_{m-1})) \leq \alpha\rho(x_{n-1}, x_{m-1}) \leq \dots \\ &\leq \alpha^n \rho(x_0, x_{m-n}) \leq \alpha^n \{\rho(x_0, x_1) + \rho(x_1, x_2) + \dots + \rho(x_{m-n-1}, x_{m-n})\} \\ &\leq \alpha^n \rho(x_0, x_1) \{1 + \alpha + \alpha^2 + \dots + \alpha^{m-n-1}\} \leq \alpha^n \rho(x_1, x_1) \frac{1}{1 - \alpha}, \end{aligned}$$

where  $\alpha < 1$ . Thus  $\rho(x_n, x_m) \rightarrow 0$  as  $n \rightarrow \infty, m > n$ . By the completeness of the space  $X$  the limit  $\lim_{n \rightarrow \infty} x_n = x$  exists. Then, by the continuity of the mapping  $g$  (cf. Proposition 3 of Sec. 1.2.4), we have  $g(x) = \lim_{n \rightarrow \infty} g(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x$ . Consequently a fixed point exists. We now prove that it is unique. If  $g(x) = x$  and  $g(y) = y$ , then  $\rho(x, y) \leq \alpha\rho(x, y)$ , i.e.,  $\rho(x, y) = 0$ , and so  $x = y$ . ■

**REMARK.** If a mapping  $g$  of a metric space  $X$  into itself has only the property that  $\rho(g(x), g(y)) < \rho(x, y)$  for any  $x, y \in X, x \neq y$ , it may be that there is no fixed point. Here is a suitable example: consider the space  $(X, \rho)$ , where  $X = [1, \infty)$  and  $\rho$  is the usual Euclidean metric. Let  $g(x) = x + \frac{1}{x}$ . Then  $\rho(g(x), g(y)) = \left| x + \frac{1}{x} - y - \frac{1}{y} \right| < |x - y|$ . There are no fixed points, since  $g(x) = x + \frac{1}{x} \neq x$  for any  $x \in X$ .

**DEFINITION 7.** The mappings  $g$  and  $g_1$  of a metric space  $(X, \rho)$  into itself are said to *commute* if the equality  $g(g_1(x)) = g_1(g(x))$  holds for every  $x \in X$ .

**THEOREM 5** (Generalization of the contraction mapping principle). *Let  $g$  and  $g_1$  be mappings of a complete metric space  $(X, \rho)$  into itself. If the mapping  $g_1$  is a contraction and the mappings  $g$  and  $g_1$  commute, then the equation  $g(x) = x$  has a solution.*

**PROOF:** By Theorem 3 there exists a unique point  $x$  such that  $g_1(x) = x$ . Apply the mapping  $g$  to both sides of this equation. Using the fact

that the mappings commute, we obtain  $g(g_1(x)) = g(x)$ , and consequently  $g_1(g(x)) = g(x)$ , i.e.,  $g_1(y) = y$ , where  $y = g(x)$ . Taking account of the fact that the mapping  $g_1$  is a contraction, and hence has only one fixed point, we find  $x = y = g(x)$ . Therefore the mapping  $g$  also has a fixed point. ■

REMARK. The  $n$ th power of the mapping  $g$  is defined as the mapping  $g^n$  obtained by  $n$  successive applications of the mapping  $g$ :

$$g^n(x) = g(g(\dots g(x))\dots), \quad x \in X.$$

It follows from Theorem 5 that if the mapping  $g$  is such that some power of it is a contraction, then the equation  $g(x) = x$  has one and only one solution. The uniqueness of the solution follows from the fact that every fixed point of the mapping  $g$  is a fixed point of the mapping  $g^n$ , while the latter is a contraction.

#### EXAMPLES

1 (*Finding the zeros of functions*). Let  $\varphi(t)$ ,  $a \leq t \leq b$ , be a real-valued function satisfying a Lipschitz condition

$$|\varphi(t_1) - \varphi(t_2)| \leq \theta |t_1 - t_2|, \quad t_1, t_2 \in [a, b], \quad 0 < \theta < 1,$$

and mapping the closed interval  $[a, b]$  into itself. If we introduce the metric space  $(X, \rho)$ , where  $X = [a, b]$  and  $\rho$  is the usual Euclidean metric on the closed interval, then the mapping  $\varphi$  is a contraction on  $X$ , and therefore the numerical sequence  $t_0, t_1 = \varphi(t_0), t_2 = \varphi(t_1), \dots$  converges to a unique root of the equation  $t = \varphi(t)$  for any  $t_0 \in [a, b]$ . The mapping  $\varphi$  is a contraction, for example, if  $|\varphi'(t)| \leq \theta < 1$  for all  $t \in [a, b]$ .

Suppose it is necessary to solve an equation of the form  $F(t) = 0$ , where  $F(t)$  is a real-valued function defined on  $[a, b]$  and  $F(a) < 0$ ,  $F(b) > 0$ ,  $0 < \theta_1 \leq F'(t) \leq \theta_2$ ,  $t \in [a, b]$ . Then if we consider the function  $\varphi(t) = t - \lambda F(t)$ ,  $\lambda \in \mathbf{R}^1$  and find a root of the equation  $\varphi(t) = t$ , the original problem will be solved. We can apply the preceding reasoning to this last equation if, for example,  $|\varphi'(t)| \leq \theta < 1$ . We have that  $1 - \lambda\theta_2 \leq \varphi'(t) \leq 1 - \lambda\theta_1$ ,  $\lambda > 0$ . It is not difficult to choose a real number  $\lambda > 0$  such that the condition  $|\varphi'(t)| \leq \theta < 1$  holds.

2 (*Solving a system of equations of the form  $y = Ax + b$* ). Let  $A = \{a_{ij}\}_{i,j=1}^n$  be a matrix,  $X$  the  $n$ -dimensional space of rows  $(x_1, \dots, x_n)$ , and  $y_i = \sum_{j=1}^n a_{ij}x_j + b_i$  a mapping of the space  $X$  into itself. The ordered set  $(x_1, x_2, \dots, x_n)$  goes to the set  $g(x) = (y_1, y_2, \dots, y_n)$ , i.e.,  $g(x) = Ax + b$ .



If the mapping  $g(x)$  is a contraction in the space  $X$  with some metric and under certain conditions, then by the preceding the vector equation  $g(x) = x$  will have one and only one solution. We shall find such conditions on the mapping  $g$  and introduce a metric on the set  $X$ , i.e., we shall form suitable metric spaces. Consider the following cases:

a) Let

$$\rho_1(x, y) = \max_{1 \leq i \leq n} |y_i - x_i|;$$

it is then easy to obtain the following estimate: if  $y' = Ax' + b$  and  $y'' = Ax'' + b$ , then

$$\rho_1(y', y'') \leq \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}| \rho(x', x''), \quad x', x'', y', y'' \in X,$$

and the condition for the mapping  $g$  to be a contraction will be satisfied if, for example,

$$\sum_{j=1}^n |a_{ij}| \leq \alpha < 1, \quad i = 1, \dots, n.$$

b) If we introduce a metric on  $X$  by the rule

$$\rho_2(x, y) = \sum_{i=1}^n |x_i - y_i|,$$

then, as is easy to verify, the mapping  $g$  will be a contraction if

$$\sum_{i=1}^n |a_{ij}| \leq \alpha < 1, \quad j = 1, \dots, n.$$

c) Finally, if the metric is defined as follows:

$$\rho_3(x, y) = \left[ \sum_{i=1}^n |x_i - y_i|^2 \right]^{1/2},$$

the mapping will be a contraction if

$$\sum_{i,j=1}^n |a_{ij}|^2 \leq \alpha < 1.$$

The conditions just written out are sufficient for the equation  $g(x) = x$  to have a unique solution, or, what is the same, for the system  $x_i = \sum_{j=1}^n a_{ij} x_j + b_i$ ,  $i = 1, \dots, n$ , to have a unique solution.

3 (*Existence and uniqueness of a solution of the Cauchy problem for a first-order differential equation*). Suppose the Cauchy problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0, \quad t, t_0 \in [a, b],$$

is given. Suppose the function  $f(t, y)$  is continuous on the set  $a \leq t \leq b$ ,  $-\infty < y < +\infty$ , and satisfies a Lipschitz condition on  $y$ , i.e.,

$$|f(t, y') - f(t, y'')| \leq K|y' - y''|$$

for any  $y'$  and  $y''$ . The existence and uniqueness of the solution of the Cauchy problem are equivalent to the existence and uniqueness of a solution of the following integral equation:

$$y(t) = y_0 + \int_{t_0}^t f(\xi, y(\xi)) d\xi.$$

Consider the mapping of the set of functions  $\{y(t)\}$  according to the rule  $g(y(t)) = y_0 + \int_{t_0}^t f(\xi, y(\xi)) d\xi$ . We introduce the space  $C[a, b]$ . Then the problem of finding a solution of the integral equation reduces to finding a fixed point of the mapping  $g$ , i.e., finding a function  $y$  such that  $g(y) = y$ . In order for such a point to exist and be unique it suffices that the mapping  $g$  be a contraction.

Since it follows from the Lipschitz condition that

$$|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|,$$

we have for  $y, z \in C[a, b]$

$$\begin{aligned} \rho(g(y), g(z)) &\leq \max_{a \leq t \leq b} \left| \int_{t_0}^t K \rho(y, z) d\xi \right| = K(b - a) \rho(y, z), \\ \rho(y, z) &= \max_{t \in [a, b]} |y(t) - z(t)|. \end{aligned}$$



Consequently the mapping is a contraction if the closed interval  $[a, b]$  is sufficiently small, i.e., if

$$K(b - a) = \theta < 1.$$

Under these conditions the theorem on the existence and uniqueness of a solution to the Cauchy problem on such an interval  $[a, b]$  holds.

4 (*Properties of a Volterra operator and powers of it*).

We shall show that some power of a mapping given by a Volterra integral operator

$$g(f(t)) = \lambda \int_a^t K(t, \xi) f(\xi) d\xi + \varphi(t),$$

where  $\lambda$  is some number and  $K(t, \xi) \in C([a, b] \times [a, b])$  and  $\varphi(t) \in C[a, b]$  are continuous functions of their arguments, is a contraction mapping in  $C[a, b]$ ,  $a \leq t \leq b$ . Let

$$M = \max_{a \leq t, \xi \leq b} |K(t, \xi)|, \quad \rho(f_1, f_2) = \max_{a \leq t \leq b} |f_1(t) - f_2(t)|.$$

Then

$$\begin{aligned} |g(f_1(t)) - g(f_2(t))| &= |\lambda| \left| \int_a^t K(t, \xi) (f_1(\xi) - f_2(\xi)) d\xi \right| \\ &\leq |\lambda| \cdot M \cdot (t - a) \cdot \rho(f_1, f_2), \end{aligned}$$

$$\begin{aligned} |g^2(f_1(t)) - g^2(f_2(t))| &= |\lambda| \left| \int_a^t K(t, \xi) (g(f_1(\xi)) - g(f_2(\xi))) d\xi \right| \\ &\leq |\lambda| \cdot M \cdot \int_a^t |\lambda| \cdot M \cdot (\xi - a) \rho(f_1, f_2) d\xi \leq \frac{|\lambda|^2 \cdot M^2 \cdot (t - a)^2}{2} \rho(f_1, f_2). \end{aligned}$$

Hence

$$|g^n(f_1) - g^n(f_2)| \leq |\lambda|^n \cdot M^n \cdot \frac{(t - a)^n}{n!} \cdot \rho(f_1, f_2).$$

It is always possible to choose  $n$  so that  $\frac{|\lambda|^n \cdot M^n \cdot (b - a)^n}{n!} < 1$  and for this  $n$  the mapping  $g^n$  is a contraction. According to the remark after Theorem 5 an integral equation of the form  $g(f) = f$  has a unique solution for any  $\lambda$ .

### 3.3. Compactness in Metric Spaces. $\varepsilon$ -Nets

We now continue our study of the properties of metric spaces by considering the properties of a compact metric space. We give the following definition.

**DEFINITION 8.** Let  $A$  be a set in a metric space  $(X, \rho)$  and  $\varepsilon$  some positive number. A set  $B$  of the space is called an  $\varepsilon$ -net for the set  $A$  (possibly  $A = X$ ) if for any point  $x \in A$  there exists a point  $y \in B$  such that  $\rho(x, y) < \varepsilon$ .

The following theorem holds.

**THEOREM 6.** Let  $X$  be a metric space. The following properties of the space  $X$  are equivalent.

- 1)  $X$  is compact;
- 2)  $X$  is complete and for any  $\varepsilon > 0$  there exists a finite  $\varepsilon$ -net in  $X$ ;
- 3) from any sequence of points of  $X$  it is possible to choose a convergent subsequence (so-called sequential compactness);
- 4) any infinite subset in  $X$  has at least one limit point (so-called countable compactness).

**PROOF:** We carry out the proof according to the "circular scheme." We begin by showing that 1) implies 2).

Let  $X$  be compact. We first show that  $X$  is complete. Let  $\{a_n\}$  be a fundamental sequence in  $X$ . Set  $A_n = \{a_n, a_{n+1}, \dots\}$  and  $B_n = \bar{A}_n$ . It follows from the compactness that  $\bigcap_{i=1}^{\infty} B_i$  is nonempty (since  $\{B_i\}$  is a centered system of closed subsets). Let  $a_0 \in \bigcap_{i=1}^{\infty} B_i$ . Then for any  $N$  and  $\varepsilon > 0$  there are points of the sequence with indices larger than  $N$  in the  $\varepsilon$ -neighborhood of the point  $a_0$ . It follows from this and from the fact that the sequence is fundamental that  $a_0$  is the limit of  $\{a_n\}$ . Therefore the space  $X$  is complete.

Now suppose there exists  $\varepsilon_0 > 0$  such that there is no finite  $\varepsilon_0$ -net in  $X$ . Take an arbitrary point  $a_1 \in X$ . There is at least one point in  $X$ , which we denote by  $a_2$ , such that  $\rho(a_1, a_2) > \varepsilon_0$ . If this were not the case, the single point  $a_1$  would be an  $\varepsilon_0$ -net in  $X$ . In exactly the same way, there is a point  $a_3 \in X$  such that  $\rho(a_1, a_3) > \varepsilon_0$  and  $\rho(a_2, a_3) > \varepsilon_0$ , etc. Let the points  $a_1, \dots, a_n$  be chosen. Then obviously there is a point  $a_{n+1} \in X$  such that  $\rho(a_i, a_{n+1}) > \varepsilon_0, i = 1, \dots, n$ . Thus we have constructed an infinite sequence of points  $a_1, a_2, \dots$ . It is easy to see that each of the sets  $A_n, \{a_n, a_{n+1}, \dots\}$  is



closed. They form a centered system with empty intersection, contradicting the compactness of  $X$ .

We now show that property 2) implies property 3). Let  $\{a_n\}$  be a sequence of points of  $X$ . We choose a finite 1-net in  $X$ , and consider the closed ball of radius 1 about each of its points. The union of these balls covers all of  $X$ , and they are finite in number. Hence at least one of them, which we call  $K_1$ , contains an infinite subsequence  $\{a_n^1\}_{n=1}^\infty$  of the sequence  $\{a_n\}$ . Choosing next a finite  $1/2$ -net and repeating the reasoning, only applying it to the sequence  $\{a_n^1\}$ , we obtain a ball  $K_2$  of radius  $1/2$  containing a subsequence  $\{a_n^2\}_{n=2}^\infty$  of the sequence  $\{a_n\}$ . Repeating the process, for each  $m$  we obtain a ball  $K_m$  of radius  $1/m$  and a subsequence  $\{a_n^m\}$  of  $\{a_n^{m-1}\}$  contained in it. Now consider the sequence  $\{b_n\}_{n=1}^\infty$ , where  $b_n = a_n^n$ . It is obvious that  $\{b_n\}$  is a subsequence of the sequence  $\{a_n\}$ . Moreover for  $m \geq n_0$  we have  $b_m \in \{a_n^{n_0}\}_{n=1}^\infty \subset K_{n_0}$ . This means that  $\{b_n\}$  is fundamental, and since  $X$  is complete, it has a limit.

The fact that property 4) follows from property 3) is obvious. We shall show finally that property 1) follows from property 4). To do this we first prove that for any  $\varepsilon > 0$  in  $X$  we can choose a finite  $\varepsilon$ -net. If this were not the case, we could construct, as in the proof that 2) follows from 1), a sequence  $a_1, a_2, \dots$  and obtain an infinite set without limit points, contrary to hypothesis. For each  $n$  we construct a finite  $1/n$ -net. Consider the union of all these nets. The set so obtained is dense in  $X$  and at most countable. Thus  $X$  is separable, and by Lemma 11 of Sec. 1.2.6 possesses a countable basis. In order to prove that a space having a countable basis is compact, it suffices to prove that any countable (not any arbitrary infinite) cover has a finite subcover (and correspondingly for centered systems of closed sets). Indeed, let  $\{U_\alpha\}$  be an arbitrary cover of the space  $X$  and  $\{V_n\}$  a countable basis of the space. Each point  $x \in X$  is contained in some  $U_\alpha$ . By definition of the basis there exists some  $V_i \in \{V_n\}$  such that  $x \in V_i \subset U_\alpha$ . If for each point  $x \in X$  we consider the neighborhood  $V_i$ , the totality of these neighborhoods will be a countable cover of  $X$ . If we can choose a finite subcover  $\{V_{i_1}, \dots, V_{i_n}\}$  from it, then, taking for each  $V_{i_j}$  a set  $U_\alpha$  containing it, we obtain a finite subcover of the original (not necessarily countable) cover.

Thus it remains only to prove that from any countable open cover of  $X$  we can choose a finite subcover. We shall prove the equivalent assertion (according to Lemma 5 of Sec. 1.2.5) for closed subsets.

Let  $\{F_n\}_{n=1}^\infty$  be a centered system of closed subsets of  $X$ . We shall show that  $\bigcap_{n=1}^\infty F_n \neq \emptyset$ . Let  $\Phi_n = \bigcap_{k=1}^n F_k$ ; it is clear that  $\Phi_n$  are closed and

nonempty since the system  $\{F_n\}$  is centered, and

$$\Phi_1 \supset \Phi_2 \supset \cdots, \quad \bigcap_{n=1}^{\infty} \Phi_n = \bigcap_{n=1}^{\infty} F_n.$$

Two cases can occur:

a) from some index on

$$\Phi_{n_0} = \Phi_{n_0+1} = \cdots = \Phi_{n_0+k} = \cdots$$

Then it is obvious that

$$\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} \Phi_n = \Phi_{n_0} \neq \emptyset;$$

b) there are infinitely many pairwise distinct  $\Phi_n$ . Here it obviously suffices to consider the case when all  $\Phi_n$  are distinct. Let  $a_n \in \Phi_n \setminus \Phi_{n+1}$ . Then the sequence  $\{a_n\}_{n=1}^{\infty}$  is an infinite set of distinct points of  $X$ , and by what we have proved, it has at least one limit point, say  $a_0$ . Since  $\Phi_n$  contains all the points  $a_n, a_{n+1}, \dots$ , it follows that  $a_0$  is a limit point of every  $\Phi_n$ , and since  $\Phi_n$  is closed (cf. Lemma 1 of Sec. 1.2.2),  $a_0 \in \Phi_n$  for any  $n$ . Therefore  $a_0 \in \bigcap_{n=1}^{\infty} \Phi_n = \bigcap_{n=1}^{\infty} F_n$ , i.e.,  $\bigcap_{n=1}^{\infty} F_n$  is nonempty. ■

It follows from the proof of the theorem that the existence of a finite  $\varepsilon$ -net is necessary and sufficient for  $X$  to be precompact.

#### EXERCISES

1. Construct a metric space  $R = (X, \rho)$  in which every one-point set is open, yet  $R$  is not complete.

2. Show that none of the conditions a), b), c) of Example 2, which are sufficient for the mapping to be a contraction, is necessary for the method of successive approximations applied in the proof of Theorem 3 to be applicable.

3. Let  $X$  be the set of continuous functions on the closed interval  $[a, b]$  and  $\rho$  a distance function defined according to the formula

$$\rho(f(t), g(t)) = \left[ \int_a^b |f(t) - g(t)|^p dt \right]^{1/p}, \quad p \geq 1,$$

(the space  $\tilde{C}^p[a, b]$ ). Show that the space  $\tilde{C}^p[a, b] = (X, \rho)$  is not complete. Verify that the spaces of the examples of Sec. 1.2.1, namely  $m$ ,  $C^n[a, b]$  ( $n \geq 1$ ), and  $l^p$  ( $p \geq 1$ ) are complete metric spaces.



4. On the line  $X = (-\infty, \infty)$  we introduce a metric according to the rule  $\rho(x, y) = \arctan |x - y|$ . Is the space  $(X, \rho)$  complete?

5. Let  $X$  be the set of bounded continuous functions on the line and  $Y$  the set of continuous functions for which  $\lim_{|z| \rightarrow \infty} f(x) = 0$ ; and  $Z$  the set of all continuous functions of compact support, i.e., continuous functions that vanish outside some interval. Are the spaces  $(X, \rho)$ ,  $(Y, \rho)$ , and  $(Z, \rho)$ , complete, where  $\rho(f, g) = \sup_t |f(t) - g(t)|$ ?

6. Let  $A$  be a set of first category in a compact metric space  $(X, \rho)$ . Prove that the complement of  $A$  in  $X$  is everywhere dense in  $(X, \rho)$ , i.e., that  $\overline{A'} = X$ .

7. Two contraction mappings  $g_1$  and  $g_2$  in a complete metric space  $(X, \rho)$  satisfy the condition  $\rho(g_1(x), g_2(x)) < 1$  for any point  $x \in X$ . Let  $\alpha = \max(\alpha_1, \alpha_2)$ , where  $\alpha_1$  and  $\alpha_2$  satisfy

$$\rho(g_1(x), g_1(y)) \leq \alpha_1 \rho(x, y), \quad \rho(g_2(x), g_2(y)) \leq \alpha_2 \rho(x, y)$$

for any  $x, y \in X$ ,  $\alpha_1 < 1$ ,  $\alpha_2 < 1$ . Then the fixed points of the mappings  $g_1$  and  $g_2$  lie in some ball of radius  $1/(1 - \alpha)$ .

8. Let  $X$  be a complete metric space and  $Y$  a subspace of it that is not closed. Prove that there exist fundamental sequences in  $Y$  that have no limit in  $Y$ , i.e.,  $Y$  is an incomplete space.

9. Prove that the closure of the set  $K$  contained in  $l^2$  consisting of elements  $x = (x_1, x_2, \dots, x_k, \dots)$  is compact if and only if all the numbers  $|x_k|$  are bounded by a fixed constant and for any  $\varepsilon > 0$  one can exhibit an index  $n$  such that  $\sum_{k=n}^{\infty} |x_k|^2 < \varepsilon$  for all elements  $x$  of  $K$ .

## 4. TOPOLOGICAL SPACES

In this section we shall study the fundamental properties of topological spaces. The material of this section is very similar to what was discussed in Sec. 1.2, and therefore we shall repeat only the basic definitions, omitting the proofs of some of the theorems since they are verbatim repetitions of the corresponding proofs in Sec. 1.2. We shall pay detailed attention only to the specific peculiarities of topological spaces.

### 4.1. Definition of a Topological Space.

#### Hausdorff Spaces. Examples.

**DEFINITION 1.** A set  $X$  is said to have the *structure of a topological space* if a system of subsets of it  $\{\Sigma\}$  is defined possessing the following properties:

- 1) the set  $X$  itself and the empty set  $\emptyset$  belong to  $\{\Sigma\}$ ;
- 2) the union of any number of sets of the system  $\{\Sigma\}$  and the intersection of any finite number of sets of the system  $\{\Sigma\}$  belong to  $\{\Sigma\}$ .

A system  $\{\Sigma\}$  satisfying conditions 1) and 2) is called a *topology* on the set  $X$ , and the sets making up  $\{\Sigma\}$  are said to be *open in this topology*.\*

Thus the pair consisting of the set  $X$  and the topology  $\{\Sigma\}$  is a topological space, which it is sometimes convenient to denote by  $T = (X, \Sigma)$ .

Definition 1 distinguishes a very general class of spaces. This class is usually restricted by adding so-called *separation axioms* to 1) and 2). Of the large number of such axioms we shall consider only the most frequently used.

**AXIOM  $T_2$  (Hausdorff):** For any distinct points  $x$  and  $y$  belonging to the set  $X$  there exist a set  $\Sigma_y$  containing the point  $y$  and a set  $\Sigma_x$  containing the point  $x$  such that both sets belong to the system  $\{\Sigma\}$  and the sets do not intersect, i.e.,  $\Sigma_x \cap \Sigma_y = \emptyset$ .

Topological spaces satisfying Axiom  $T_2$  (Hausdorff's axiom) are called *Hausdorff spaces*.

**AXIOM  $T_1$ .** For any two distinct points  $x$  and  $y$  belonging to the set  $X$  there exists a set  $\Sigma_x$  belonging to the system  $\{\Sigma\}$  and containing the point  $x$  but not the point  $y$ , and there exists a set  $\Sigma_y$  of the system  $\{\Sigma\}$  containing the point  $y$  but not the point  $x$ .

Topological spaces satisfying Axiom  $T_1$  are called  *$T_1$ -spaces*.

It is clear that if Axiom  $T_2$  holds, then Axiom  $T_1$  holds also, i.e., the class of topological spaces satisfying axioms 1), 2), and  $T_2$  is smaller than the class of topological spaces satisfying axioms 1), 2), and  $T_1$ , which in turn is smaller than the class of topological spaces satisfying axioms 1) and 2).

An example of a space satisfying axioms 1), 2), and  $T_1$  but not axiom  $T_2$  is the following. The set  $X$  consists of the points of the closed interval  $[0, 1]$  and the open sets are defined to be  $X$ ,  $\emptyset$ , and  $\Sigma_{\alpha_n} = [0, 1] \setminus \{\alpha_n\}$ , where  $\{\alpha_n\}$  are arbitrary finite or countable sets in the closed interval  $[0, 1]$ . It is obvious that axioms 1), 2), and  $T_1$  holds, but  $T_2$  does not hold.

Not every topological space satisfies the axiom  $T_1$ . Here is the traditional example. The set  $X = \{a, b\}$  consists of two points. We define the topology by giving the open sets, which we take to be  $X$ ,  $\emptyset$ , and  $\{b\}$ . Axioms 1) and 2) hold, but  $T_1$  does not.

In the next section we shall become acquainted with other separation axioms of no less importance.

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\*The set complementary to an open set is called *closed* in the topology.



We now give the most commonly-encountered examples of topological spaces.

### EXAMPLES

1. Consider an arbitrary metric space  $R = (X, \rho)$ . By Lemma 1 of Sec. 1.2 the open sets satisfy properties 1) and 2) of Definition 1 of a topological space. Axiom  $T_2$  of Definition 1 also holds in a metric space: if  $x \neq y$ , then  $\rho(x, y) = a > 0$  and the balls  $O(x, a/3)$  and  $O(y, a/3)$  are open sets in  $R = (X, \rho)$  such that  $O(x, a/3) \cap O(y, a/3) = \emptyset$ .

Thus every metric space  $R = (X, \rho)$  is also a Hausdorff topological space  $T = (X, \Sigma)$ , where  $\{\Sigma\}$  is the system of open sets in  $R = (X, \rho)$ .

2. Consider an arbitrary set  $X$ . We assign to the system  $\{\Sigma\}$  only the whole set  $X$  and the empty set  $\emptyset$ . Axioms 1) and 2) obviously hold. However Axioms  $T_1$  and  $T_2$  do not hold.\* Such a topology is called the *indiscrete topology*.

3. Let  $X$  be an arbitrary set. We assign to the system  $\{\Sigma\}$  all the subsets of  $X$ . It is easy to verify that  $\{\Sigma\}$  is a Hausdorff topology. This topology is called the *discrete topology*.

We give the following definition.

**DEFINITION 2.** A *neighborhood* of a point  $x$  belonging to a topological space  $T = (X, \Sigma)$  is an open set containing the point  $x$ . A *neighborhood* of a subset of  $X$  (possibly  $X$  itself) is any open set containing the given subset (or  $X$ ). We shall denote a neighborhood of the point  $x$  by  $\Sigma_x$ .

We shall assume that among all the neighborhoods of a point  $x$  belonging to the topological space  $T = (X, \Sigma)$  certain ones are distinguished in such a way that for any point  $x$  and any neighborhood  $\Sigma_x$  of it there exists a neighborhood  $\Sigma_x^1$  of the distinguished system such that  $x \in \Sigma_x^1 \subset \Sigma_x$ .

**DEFINITION 3.** The system of distinguished neighborhoods  $\{\Sigma_x^1\}$  is called a *basis* of the topology of  $X$ .\*\*

We have the following lemma, which gives a convenient method of defining a topology.

**LEMMA 1.** Let  $X$  be an arbitrary set. For each point  $x$  define certain subsets  $\Sigma_x$ , called "*neighborhoods*" of the point  $x$  and satisfying the following conditions:

\*Unless  $X$  consists of a single point. Tr.

\*\*If  $x$  is fixed, the system  $\{\Sigma_x^1\}$  is called a *neighborhood basis* at the point  $x$ .

a) each point has at least one neighborhood and belongs to each of its neighborhoods;

b) the intersection of two neighborhoods of a point contains a third neighborhood of that point;

c) for any neighborhood  $\Sigma_x$  of a point  $x \in X$  and any point  $y \in \Sigma_x$ , there exists a neighborhood  $\Sigma_y$  of  $y$  such that  $\Sigma_y \subset \Sigma_x$ .

Then if we assign the empty set and all the neighborhoods  $\Sigma_x$  and all unions of such neighborhoods to the system  $\{\Sigma\}$ , a topology will be defined on  $X$  and  $T = (X, \Sigma)$  will be a topological space in which the system of neighborhoods is a basis. Conversely every topological space can be obtained in this way.

PROOF: We shall verify that axioms 1) and 2) of a topological space hold. The fact that the whole set  $X$  belongs to  $\{\Sigma\}$  is obvious, and  $\emptyset$  was assigned to  $\{\Sigma\}$  by definition. Hence axiom 1) holds.

To verify axiom 2) we need only verify that  $\Sigma_1 \cap \Sigma_2 \in \{\Sigma\}$  if  $\Sigma_1 \in \{\Sigma\}$  and  $\Sigma_2 \in \{\Sigma\}$ . Consequently we need to establish that  $\Sigma_1 \cap \Sigma_2$  can be obtained as the union of some neighborhoods, i.e., that for any point  $x \in \Sigma_1 \cap \Sigma_2$  there exists a neighborhood  $\Sigma_x \subset \Sigma_1 \cap \Sigma_2$ . But  $\Sigma_1$  and  $\Sigma_2$  belong to  $\{\Sigma\}$ , and so there are neighborhoods  $\Sigma_x^1 \subset \Sigma_1$  and  $\Sigma_x^2 \subset \Sigma_2$ . By condition b) the intersection of these two neighborhoods contains some neighborhood  $\Sigma_x$  of the point  $x$ , and this neighborhood is obviously contained in  $\Sigma_1 \cap \Sigma_2$ .

Conversely if a topological space  $T = (X, \Sigma)$  is given, we can take all the sets of the system  $\{\Sigma\}$  containing the point  $x$  as the neighborhoods of  $x$  satisfying a)-c). ■

Using this lemma we give two more examples of Hausdorff topological spaces.

#### EXAMPLES

1. We take  $X$  to be the plane  $\mathbf{R}^2$ . We obtain a neighborhood of any point  $x \in X$  by removing from any disk centered at  $x$  all the points different from  $x$  lying on the vertical diameter of the disk. The topology so obtained is Hausdorff.

2. Let  $X$  be the closed interval  $[0, 1]$ . Let the neighborhoods of all points except 0 be the ordinary neighborhoods. Neighborhoods of 0 are taken to be half-open intervals  $[0, \alpha]$ ,  $\alpha > 0$ , with the points  $1/n$  removed,  $n$  being natural numbers. This, as is easy to see, is an example of a Hausdorff topological space.

Let  $T = (X, \Sigma)$  be a topological space and  $Y$  a subset of  $X$ . Then on the subset  $Y$  we can consider the trace of the system  $\{\Sigma\}$ , i.e., the sets of



the form  $\{\Sigma_Y\} = \{Y \cap \Sigma_\alpha\}$ , with  $\Sigma_\alpha \in \{\Sigma\}$ . It is easy to see that this defines a topology on  $Y$ , and so  $Y$  becomes a topological space in its own right. The space  $T_Y = (Y, \Sigma_Y)$  is called a *subspace* of the space  $T$ . The topology defined by the system  $\{\Sigma_Y\} = \{Y \cap \Sigma_\alpha\}$ ,  $\Sigma_\alpha \in \{\Sigma\}$  is called the *induced topology*.

Just as in the case of metric spaces, a space  $T = (X, \Sigma)$  is called *connected* if it cannot be represented as the union of two nonempty disjoint open sets. A subset  $Y$  of a topological space  $T$  is connected if  $Y$  is connected as a subspace of  $T$ :  $(Y, \Sigma_Y) \subset (X, \Sigma)$ .

## 4.2. A Remark on Topological Spaces

After open sets have been introduced one can introduce into topological spaces all the concepts introduced in Sec. 1.2 for metric spaces. Thus the definition of a limit point of a set  $Y$  carries over verbatim (cf. Definition 4 of Sec. 1.2), as does the definition of an interior point\* (cf. Definition 5 of Sec. 1.2), the definition of a closed set (Definition 6 of Sec. 1.2), the definition of the closure of a set (Definition 7 of Sec. 1.2), and the definition of a dense set and an everywhere dense set (Definition 10 of Sec. 1.2). The concepts of nowhere dense and perfect sets given in Sec. 1.2 for metric spaces are preserved completely. Just as in the case of metric spaces there is the important concept of continuous mapping in the case of topological spaces (Definition 12 of Sec. 1.2), as well as the concept of a homeomorphic mapping (Definition 13 of Sec. 1.2) and compact set (Definition 14 of Sec. 1.2). The concept of a centered system is introduced for topological spaces just as was done in Sec. 1.2 for metric spaces (Definition 15 of Sec. 1.2). So also are the concepts of a basis for the topology (Definition 16 of Sec. 1.2) and a topological space having a countable basis. Topological spaces with a countable base are said to satisfy the second axiom of countability.

The reader will easily be able to formulate these definitions for the case of topological spaces; to do so one need only replace the expression *metric space* by the expression *topological space* in the corresponding definitions of Sec. 1.2.

According to these definitions the basic propositions proved for metric spaces in Sec. 1.2 remain valid in the case of a topological space. This is completely natural since the proofs of these propositions mainly use the concept of open and closed set and do not depend directly on the metric introduced.

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\*The set of all interior points of a set  $Y$  is called the *interior* of  $Y$  and denoted  $\overset{\circ}{Y}$ . The interior  $\overset{\circ}{Y}$  is obviously the union of all open sets contained in  $Y$ .

Thus Lemma 1 of Sec. 1.2, which asserts that the union of an arbitrary number of open sets is open and the intersection of a finite number of open sets is open is an axiom for a topological space. The assertion of Lemma 2 of Sec. 1.2, including also the proofs of the properties of the closure operation, carries over completely.

The concept of convergent sequence also carries over to topological spaces (Definition 11 of Sec. 1.2). To be specific a sequence  $\{a_n\}$  of points of a topological space is said to converge to the point  $a$  of this space if any neighborhood of the point  $a$  contains all but a finite number of terms of the sequence  $\{a_n\}$ . If the sequence  $\{a_n\}$  converges to the point  $a$ , we write that  $a_n \rightarrow a$  as  $n \rightarrow \infty$  or  $\lim_{n \rightarrow \infty} a_n = a$ .

However this concept does not play as large a role in topological spaces as it does in metric spaces. In fact Lemma 3 of Sec. 1.2 asserted that in a metric space the point  $a$  belongs to the closure  $\bar{A}$  of a set  $A$  if and only if there exists a sequence  $\{a_n\}$  of points of the set  $A$  converging to  $a$ . In a topological space this assertion may be false. (We recall that in the proof of this proposition in metric spaces we constructed a nested sequence of balls  $O(a, 1/n)$ .)

It is possible to distinguish a class of topological spaces having an analogous property.

A topological space is said to satisfy the *first axiom of countability* (or to be *first-countable*) if for any point  $a$  of the space there exists a countable system of neighborhoods  $\{\Sigma_a^n\}$  of  $a$  such that for any open set  $\Sigma_a$  containing the point  $a$  there exists a neighborhood  $\Sigma_a^{n_0} \subset \Sigma_a$ . Such a system of neighborhoods is called a *basis* of the neighborhood system of the point  $a$  (cf. the footnote following Definition 3 above).

The first axiom of countability clearly holds in a metric space.

The following proposition holds in a first-countable topological space: a point  $a \in T$  belongs to the closure  $\bar{A}$  of a set  $A$  if and only if there exists a sequence  $\{a_n\}$  of points of the set  $A$  converging to  $a$ .

The proof of this proposition is similar to the proof of the corresponding proposition for the case of metric spaces. The sequence of balls  $O(a, 1/n)$  must be replaced by a sequence of neighborhoods of the system  $\{\Sigma_a^n\}$ , and we can always assume that  $\Sigma_a^{n+1} \subset \Sigma_a^n$ . Otherwise one need only replace  $\Sigma_a^n$  by  $\bigcap_{k=1}^n \Sigma_a^k$ .

Proposition 1 and Proposition 2 of Sec. 1.2 carry over completely. The assertion of Lemma 4 of Sec. 1.2—the criterion for continuity of a mapping—also carries over.

The criterion for compactness in terms of centered systems of closed subsets (Lemma 5 of Sec. 1.2) carries over completely to the case of topolog-



ical spaces, as do Proposition 4 and the assertions of Lemmas 6, 7, 8, and 9 on the properties of a compact set and continuous functions on it.

It follows from Lemma 10 of Sec. 1.2, which is valid for topological spaces also, that the system  $\{\Sigma_\alpha\}$  forms a basis if and only if it is a neighborhood basis of the space. Thus these two concepts are equivalent.

We remark that a topological space may fail to be second-countable even when it is first-countable and contains a countable dense set. However, if there is a countable basis for the topology, then the topological space is separable and satisfies the first axiom of countability (compare with Lemma 11 of Sec. 1.2). Just as in the case of metric spaces (cf. Definition 17 of Sec. 1.2) a topological space is called *second-countable* if there exists a countable basis for its topology.

### EXAMPLES

1. A set  $A$  of the topological space  $T$  is closed if and only if it coincides with its closure  $\bar{A}$ , i.e.,  $A = \bar{A}$  (cf. Example 2 of Sec. 1.2).

2. A set in a topological space is called a  $G_\delta$ -set if it is the intersection of a countable number of open sets. The complement of a  $G_\delta$ -set is called an  $F_\sigma$ -set. For example the set of rational numbers in  $\mathbf{R}^1$  is an  $F_\sigma$ -set. The set of irrational numbers is a  $G_\delta$ -set.

3. The closure  $\bar{A}$  of a set  $A$  in a topological space consists of the points that are either limit points of the set  $A$  or elements of  $A$  (compare with Proposition 2 of Sec. 1.2).

4. An important class of sets in a topological space is the so-called *Borel sets*, which are obtained from the open (or closed) sets using a countable number of operations, each of which is either union, intersection, or complementation.

The family  $\{B\}$  of Borel sets in a topological space is the smallest family of sets satisfying the following conditions:

- a) every open set belongs to  $\{B\}$ ;
- b) if a set  $A$  belongs to  $\{B\}$ , then  $A' \in \{B\}$ , where  $A'$  is the complement of  $A$ ;
- c) if the set  $A_n$  belongs to  $\{B\}$ , then  $\bigcup_{n=1}^{\infty} A_n \in \{B\}$ .

### EXERCISES

1. A mapping  $g : A \rightarrow B$  of a directed set  $A$  into the set  $B$  is called a *generalized sequence* (or *net*) in  $B$ . A generalized sequence  $g : A \rightarrow X$  in a topological space  $T = (X, \Sigma)$  is said to *converge* to the point  $x \in X$  if for

any neighborhood  $\Sigma_x$  of the point  $x$  there exists a point  $a_0 \in A$  such that the relations  $a \succ a_0$ ,  $a \in A$ , imply that  $g(a) \in \Sigma_x$ . In this case we say that the limit of  $g$  over  $A$  exists and is  $x$ , i.e.,

$$\lim_A g(a) = x.$$

Prove that in a Hausdorff space  $T = (X, \Sigma)$  each generalized sequence has at most one limit.

2. A filter  $\{F\}$  (cf. Sec. 1.1) of subsets of a topological space  $T = (X, \Sigma)$  is said to converge to the point  $x \in X$  if each neighborhood of the point  $x$  belongs to  $\{F\}$ .

Prove that in a Hausdorff space a filter can converge to only one point, called the *limit* of the filter.

3. Prove that a necessary and sufficient condition for a topological space to be compact is that each ultrafilter of subsets of the space converge to some point of the space.

4. A mapping  $g$  of one topological space  $T = (X, \Sigma)$  into another  $T_0 = (X_0, \Sigma_0)$  is called *closed* if the relation  $A' \in \Sigma$  implies that  $(g(A))' \in \Sigma_0$ . (Here  $A'$  is the complement of the set  $A$ .) Give examples of topological spaces and a continuous mapping of one into the other that is not closed.

5. A mapping  $g$  of one topological space  $T = (X, \Sigma)$  into another  $T_0 = (X_0, \Sigma_0)$  is called *open* if the relation  $A \in \Sigma$  implies that  $g(A) \in \Sigma_0$ . Give examples of topological spaces and a continuous mapping of one into the other that is not open.

6. Let  $T = (X, \Sigma)$  be a topological space and  $T_0 = (Y, \Sigma_Y)$  a subspace of it:  $Y \subset X$ . Let the system  $\{\Sigma_\alpha\}$  be a basis for the topology of  $T$ . Denote by  $\{\Sigma_\alpha^0\}$  the collection of sets of the form  $\Sigma_\alpha \cap Y$ . Then  $\{\Sigma_\alpha^0\}$  is a basis of the topology in  $T_0$ . Prove this.

7. Prove that in a  $T_1$ -space (consequently also in a Hausdorff space) each point is a closed set.

## 5. PROPERTIES OF TOPOLOGICAL SPACES

In this section we discuss the fundamental properties of topological spaces. Many of them are connected with the new separation axioms given below.

### 5.1. Regular, Completely Regular, and Normal Spaces

AXIOM  $T_3$ : For any point  $a$  and any closed set  $F$  not containing  $a$  there exist two disjoint open sets  $\Sigma_a$  and  $\Sigma$  such that  $a \in \Sigma_a$  and  $F \subset \Sigma$ .



Topological spaces satisfying the axioms  $T_1$  and  $T_3$  are called *regular*.

AXIOM  $T_{3\frac{1}{2}}$ : For any point  $a$  and any closed set  $F$  not containing the point  $a$  there exists a continuous numerical-valued function  $f$  defined on the space such that  $0 \leq f(x) \leq 1$  for all  $x$ ,  $f(a) = 0$ , and  $f(x) = 1$  for all  $x \in F$ .

Topological spaces satisfying the axioms  $T_1$  and  $T_{3\frac{1}{2}}$  are called *completely regular* or *Tikhonov\** spaces.

AXIOM  $T_4$ : For any two disjoint closed sets  $F_1$  and  $F_2$  there exist two disjoint open sets  $\Sigma_1$  and  $\Sigma_2$  such that  $F_1 \subset \Sigma_1$  and  $F_2 \subset \Sigma_2$ .

Topological spaces satisfying axioms  $T_1$  and  $T_4$  are called *normal* spaces.

We remark that neither the Hausdorff axiom nor even Axiom  $T_1$  follows from any of the axioms  $T_3$ ,  $T_{3\frac{1}{2}}$ , and  $T_4$ .

Examples of nonregular Hausdorff spaces are Examples 1 and 2 at the end of Sec. 1.4.1. Every completely regular space is obviously regular. Indeed consider a completely regular space. Let  $f$ , the point  $a$ , and the set  $F$  be those in the statement of Axiom  $T_{3\frac{1}{2}}$ . Let  $I_0 = [0, 1/2)$  and  $I_1 = (1/2, 1]$ . Then the sets  $\Sigma_a = f^{-1}(I_0)$  and  $\Sigma = f^{-1}(I_1)$  are open, since  $f$  is a continuous mapping, and they are obviously disjoint and satisfy  $a \in \Sigma_1$ ,  $F \subset \Sigma$ .

Consequently for any point  $a$  and any closed set  $F$  not containing it we have constructed two disjoint open sets  $\Sigma_1$  and  $\Sigma$  such that  $a \in \Sigma_1$  and  $F \subset \Sigma$ , i.e., we have shown that a completely regular space is regular.

It turns out that normal topological spaces are completely regular. This fact follows easily from the lemma proved below.

LEMMA (Uryson). For any two disjoint closed sets  $F_1$  and  $F_2$  of a normal topological space  $T = (X, \Sigma)$  there exists a continuous numerical-valued function  $f$  defined on  $T$  such that  $0 \leq f(x) \leq 1$  for all  $x \in X$ ,  $f(x) = 0$  for  $x \in F_1$ , and  $f(x) = 1$  for  $x \in F_2$ .

PROOF: We shall construct a sequence of open sets  $\Sigma(r)$  corresponding to rational numbers of the form  $r = k/2^n$ ,  $k = 0, 1, \dots, 2^n$ ,  $n$  an integer,  $n \geq 0$ , having the properties

- 1)  $F_1 \subset \Sigma(0)$ ,  $F_2 = \Sigma'(1)$  (the prime denotes complementation) and
- 2)  $\bar{\Sigma}(r) \subset \Sigma(r')$  for  $r < r'$ .

We shall carry out the construction by induction on  $n$ . Let  $n = 0$ . Then the system  $\Sigma(r)$  must contain only two sets. Since the space  $T$  is normal, there exist disjoint open sets  $\Sigma_1$  and  $\Sigma_2$  such that  $F_1 \subset \Sigma_1$  and  $F_2 \subset \Sigma_2$ . Then we define the desired sets by  $\Sigma(0) = \Sigma_1$  and  $\Sigma(1) = F_2'$ .

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\*In older textbooks, Tikhonov is spelled Tychonoff. Tr.

Now suppose that the sets  $\Sigma(r)$  are constructed for the numbers  $r = k/2^{n-1}$ ,  $k = 0, 1, \dots, 2^{n-1}$ , and conditions 1) and 2) are satisfied in this case. Take some odd integer value  $k > 0$ ,  $k + 1 \leq 2^n$ . Then  $\Sigma(\frac{k-1}{2^n}) \subset \Sigma(\frac{k+1}{2^n})$ , since the numbers  $\frac{k-1}{2^n}$  and  $\frac{k+1}{2^n}$  have the form  $\frac{k'}{2^{n-1}}$ , where  $0 \leq k' \leq 2^{n-1}$ . Since  $T$  is a normal space and  $\overline{\Sigma(\frac{k-1}{2^n})} \cap \Sigma'(\frac{k+1}{2^n}) = \emptyset$  and these sets are closed, there exist open sets  $G$  and  $G_1$  such that  $\overline{\Sigma(\frac{k-1}{2^n})} \subset G$ ,  $\Sigma'(\frac{k+1}{2^n}) \subset G_1$  and  $G_1 \cap G = \emptyset$ . Then  $G'_1$  is closed and  $G \subset G'_1 \subset \Sigma(\frac{k+1}{2^n})$ , i.e.,  $\overline{G} \subset \Sigma(\frac{k+1}{2^n})$ . Setting  $\Sigma(\frac{k}{2^n}) = \Sigma(r) = G$ , we complete the inductive construction.

We now define the function  $f(x)$  by setting

$$f(x) = \begin{cases} 0, & \text{if } x \in \Sigma(0), \\ \sup_{x \notin \Sigma(r)} r, & \text{if } x \in \Sigma'(0). \end{cases}$$

Then by condition 1) we have  $f(x) = 0$  if  $x \in F_1$  and  $f(x) = 1$  if  $x \in F_2$ . Indeed if  $x \in F_2$  then  $x \in \Sigma'(1)$ , i.e.,  $x \notin \Sigma(1)$ , so that  $x \in \Sigma'(r)$  for  $r < 1$ . In addition if  $x \in F_2$ , then  $x \notin \Sigma(0) = \Sigma_1$  so that  $x \in \Sigma'(0)$  and the indicated upper bound is 1, since it is taken over all the numbers  $r$  indicated above.

We shall now prove that the function  $f(x)$  is continuous. Take an arbitrary  $x_0 \in X$  and an arbitrary integer  $n > 0$ . We choose  $r$  such that  $f(x_0) < r < f(x_0) + 2^{-n-1}$ . Let  $\Sigma = \Sigma(r) \cap \overline{(\Sigma(r - 2^{-n}))}'$ , where we assume that  $\Sigma(s) = \emptyset$  for  $s < 0$  and  $\Sigma(s) = X$  for  $s > 1$ . The open set  $\Sigma$  contains the point  $x_0$  since  $f(x_0) < r$  and so  $x_0 \in \Sigma(r)$ ; in exactly the same way it follows from the inequality  $(r - 2^{-n-1}) < f(x_0)$  that  $x_0 \in \Sigma(r - 2^{-n-1}) \subset \overline{(\Sigma(r - 2^{-n}))}'$ . Further if  $x \in \Sigma$ , then  $x \in \Sigma(r)$ , and therefore  $f(x) \leq r$ . Moreover  $x \in \overline{(\Sigma(r - 2^{-n}))}' \subset \Sigma'(r - 2^{-n})$  and consequently  $r - 2^{-n} \leq f(x)$ . Therefore if  $x \in \Sigma$  it follows that  $|f(x) - f(x_0)| \leq 1/2^n$ . Consequently for any neighborhood  $\Sigma_{f(x_0)}$  (on the closed interval  $[0, 1]$ ) there exists a neighborhood  $\Sigma$  of the point  $x_0$  such that  $f(\Sigma) \subset \Sigma_{f(x_0)}$ , i.e., the function  $f(x)$  is continuous on  $T$ . ■

**COROLLARY.** *A normal topological space is completely regular.*

To prove this one need only use the lemma and remark that in a  $T_1$  topological space every point is a closed set (Exercise 7 of Sec. 1.4.2).

## 5.2. Regular Spaces with a Countable Basis. Tikhonov's Theorem

In this section we prove a theorem establishing a connection between the separation axioms and the countability axioms (cf. Sec. 1.4.2).



**THEOREM 1 (Tikhonov).** *A regular topological space whose topology has a countable basis is normal.*

**PROOF:** Let  $T = (X, \Sigma)$  be a regular topological space whose topology has a countable basis  $\{\Sigma_n\}_{n=1}^{\infty}$ , and let  $F_1$  and  $F_2$  be two disjoint closed sets. For any point  $x \notin F_2$  there exists a neighborhood  $U_x$  such that  $\bar{U}_x \cap F_2 = \emptyset$ . This follows from the fact that in a regular space every neighborhood of a point contains the closure of some neighborhood of the point.\* From the cover  $\{U_x : x \in F_1\}$  of the set  $F_1$  we can choose a countable subcover, i.e., in the set  $F_1$  there exists a countable system of points  $x_{1,1}, x_{1,2}, \dots, x_{1,p}, \dots$ , such that

$$F_1 \subset \bigcup_{k=1}^{\infty} U_{x_{1,k}}.$$

Here  $\bar{U}_{x_{1,k}} \cap F_2 = \emptyset$ . Similarly we can choose a countable cover  $\{U_{x_{2,k}}\}_{k=1}^{\infty}$  of the set  $F_2$  such that  $\bar{U}_{x_{2,k}} \cap F_1 = \emptyset$  for any  $k$ .

We now define the sets  $V_{1,n}$  and  $V_{2,n}$  inductively for any  $n \geq 1$  by setting

$$V_{1,n} = U_{x_{1,n}} \setminus \bigcup_{k=1}^n \bar{U}_{x_{2,k}}, \quad V_{2,n} = U_{x_{2,n}} \setminus \bigcup_{k=1}^n \bar{U}_{x_{1,k}}.$$

It is easy to see that the sets  $V_{1,n}$  and  $V_{2,m}$  are disjoint for any  $n$  and  $m$ . Indeed if  $n \leq m$ , then

$$V_{1,n} \cap V_{2,m} \subset U_{x_{1,n}} \cap (U_{x_{2,m}} \setminus \bar{U}_{x_{1,n}}) = \emptyset.$$

(If  $n > m$  the verification is similar.)

Consequently the sets

$$V_1 = \bigcup_{n=1}^{\infty} V_{1,n}, \quad V_2 = \bigcup_{n=1}^{\infty} V_{2,n}$$

are also disjoint. On the other hand they contain respectively the sets  $F_1$  and  $F_2$ . Thus we have proved that any two disjoint closed sets of the space  $T$  can be enclosed in two disjoint open sets, so that the space  $T$  is normal. ■

### 5.3. Compact Hausdorff and Normal Spaces

The following theorem establishes a connection between compactness and normality for Hausdorff spaces.

\*Let  $\Sigma$  be some neighborhood of the point  $a$ . Then  $A = X \setminus \Sigma$  is a closed set, and there exists an open set  $\Sigma_a$  containing the point  $a$  and an open set  $G$  containing  $A$  such that  $\Sigma_a \cap G = \emptyset$ . Then  $\bar{\Sigma}_a \subset X \setminus A = \Sigma$ , which was to be proved.

**THEOREM 2.** *A compact Hausdorff space is normal.*

**PROOF:** Let  $T = (X, \Sigma)$  be a compact Hausdorff space. We first show that it is regular. Let  $a$  and  $F$  be a point and a closed set not containing the point:  $a \notin F$ . For any point  $x \in F$  and the point  $a$  there exist neighborhoods  $\Sigma_x$  and  $\Sigma_a^x$  such that  $\Sigma_x \cap \Sigma_a^x = \emptyset$ . A closed subspace  $F$  of a compact space is itself compact (cf. for example, Lemma 6 of Sec. 1.2). We choose a finite subcover  $\Sigma_{x_1}, \Sigma_{x_2}, \dots, \Sigma_{x_n}$  of the cover of  $F$  by the open sets  $\{\Sigma_x\}$ . Let  $\Sigma_1 = \bigcup_{k=1}^n \Sigma_{x_k}$ , and denote the intersection of the neighborhoods  $\Sigma_a^{x_k}$  of the

point  $a$  corresponding to them by  $\Sigma_2 = \bigcap_{k=1}^n \Sigma_a^{x_k}$ . Then  $a \in \Sigma_2$ ,  $F \subset \Sigma_1$  and  $\Sigma_2 \cap \Sigma_1 = \emptyset$ . Consequently the space  $T$  is regular.

Now let  $F_1$  and  $F_2$  be two closed sets such that  $F_1 \cap F_2 = \emptyset$ . By the regularity of the space, for any point  $x \in F_1$  there exist open sets  $\Sigma_x$  and  $G_x$  such that  $x \in \Sigma_x$ ,  $F_2 \subset G_x$  and  $\Sigma_x \cap G_x = \emptyset$ . From the cover of the compact space  $F_1$  by open sets  $\{\Sigma_x\}$  we choose a finite subcover  $\{\Sigma_{x_k}\}_{k=1}^m$  and we use the notation  $\Sigma^1 = \bigcup_{k=1}^m \Sigma_{x_k}$ . We denote the intersection of the corresponding sets  $G_{x_k}$  by  $\Sigma^2 = \bigcap_{k=1}^m G_{x_k}$ . Then  $F_1 \subset \Sigma^1$ ,  $F_2 \subset \Sigma^2$ , and  $\Sigma^1 \cap \Sigma^2 = \emptyset$ , i.e., the space  $T = (X, \Sigma)$  is normal. ■

#### 5.4. Metric and Topological Spaces

We have already mentioned that in any metric space  $R = (X, \rho)$  the open sets defined by the distance function  $\rho$ , satisfy axioms 1)–2), defining a topology on the set  $X$ , and the Hausdorff axiom  $T_2$  as well. Thereby a metric space is a Hausdorff topological space. (cf. the example of Sec. 1.4.1).

The first axiom of countability holds in any metric space (cf. Sec. 4.2), as does the following theorem.

**THEOREM 3.** *A metric space is a normal topological space.*

**PROOF:** Let  $R(X, \rho)$  be a metric space and  $F_1$  and  $F_2$  two closed sets with  $F_1 \cap F_2 = \emptyset$ . Let  $x \in F_1$  and  $y \in F_2$ . Let  $\Sigma_x$  be the ball with center at  $x$  and radius  $\rho(x, F_2)/3$  (cf. Exercise 1 of Sec. 1.2) and  $\Sigma_y$  the ball with center at  $y$  of radius  $\rho(y, F_1)/3$ . It is obvious that the quantities  $\rho(x, F_2)$  and  $\rho(y, F_1)$  are positive because  $F_1$  and  $F_2$  are closed. Then  $\Sigma_1 = \bigcup_{x \in F_1} O\left(x, \frac{\rho(x, F_2)}{3}\right)$

and  $\Sigma_2 = \bigcup_{y \in F_2} \left(y, \frac{\rho(y, F_1)}{3}\right)$  are open sets containing  $F_1$  and  $F_2$  respectively.

Then  $\Sigma_1 \cap \Sigma_2 = \emptyset$ . Indeed, if  $z \in \Sigma_1 \cap \Sigma_2$  there would be a point  $x_0 \in F_1$  such that  $\rho(x_0, z) < \rho(x_0, F_2)/3$  and a point  $y_0 \in F_2$  such that  $\rho(z, y_0) <$



$\rho(y_0, F_1)/3$ . Without loss of generality assume  $\rho(x_0, F_2) \leq \rho(y_0, F_1)$ . Then by the triangle inequality  $\rho(x_0, y_0) \leq \rho(x_0, z) + \rho(z, y_0) < \rho(y_0, F_1)$ , i.e.,  $x_0 \in O(y_0, \rho(y_0, F_1))$ , contradicting the definition of  $\rho(y_0, F_1)$ . ■

Thus a metric space, regarded as a topological space, satisfies the first axiom of countability and is normal. It follows from this that in no topological space that does not satisfy the first axiom of countability or is not normal can the open sets be defined in terms of a metric.

**DEFINITION 3.** A topological space  $T = (X, \rho)$  is called *metrizable* if its topology  $\{\Sigma\}$  can be defined using some metric.

The first axiom of countability and normality are necessary conditions for metrizability. Theorems giving sufficient conditions for metrizability are called *metrization theorems*.

### 5.5. Cartesian Products of Topological Spaces

We now turn to the study of an important concept—the Cartesian product of topological spaces.

**DEFINITION 4.** The *Cartesian product* of the Hausdorff topological spaces  $T_\alpha = (X_\alpha, \Sigma_\alpha)$ ,  $\alpha \in \Omega$ , where  $\Omega$  is some set, is defined to be the pair

$$T = (X, \Sigma) = \prod_{\alpha \in \Omega} T_\alpha, \quad \text{where } X = \prod_{\alpha \in \Omega} X_\alpha,$$

and the system  $\{\Sigma\}$  is the Hausdorff topology defined as follows: the open sets are all unions of sets of the form  $\prod_{\alpha \in \Omega} \Sigma_\alpha$ , where  $\Sigma_\alpha$  are open sets in the spaces  $T_\alpha$  coinciding with  $X_\alpha$  for all but a finite number of indices  $\alpha$ . This topology is called the *Tikhonov topology*.

Let us verify that we have indeed defined a Hausdorff topology. Since the verification of the first two axioms defining a topology is simple, we shall verify only Axiom  $T_2$ . Let  $f_1(\alpha)$  and  $f_2(\alpha)$ ,  $\alpha \in \Omega$ , be two distinct points of the space  $T = (X, \Sigma) = \prod_{\alpha \in \Omega} T_\alpha$ ,  $T_\alpha = (X_\alpha, \sigma_\alpha)$ ,  $X = \prod_{\alpha \in \Omega} X_\alpha$ . Then there exists a space  $T_{\alpha_1}$ , such that  $f_1(\alpha_1) \neq f_2(\alpha_1)$ . Since there exist disjoint neighborhoods  $\Sigma_{\alpha_1}^1$  and  $\Sigma_{\alpha_1}^2$  of the points  $f_1(\alpha_1)$  and  $f_2(\alpha_1)$ , the neighborhoods  $\Sigma_1 = f^{-1}(\Sigma_{\alpha_1}^1)$  and  $\Sigma_2 = f^{-1}(\Sigma_{\alpha_1}^2)$  in the space  $T$  obviously do not intersect. Here  $f^{-1}(\Sigma_{\alpha_1}^i)$  are the preimages under the “projection”  $f: T \rightarrow T_{\alpha_1}$  of the sets  $\Sigma_{\alpha_1}^i$ ,  $i = 1, 2$ .

We shall now study the product of a finite or countable number of metric spaces and prove the following theorem.

**THEOREM 4.** *The product of a finite or countable number of metric spaces is metrizable.*

PROOF: We carry out the proof for the more complicated countable case. Let  $(X_n, \rho_n)$ ,  $n = 1, 2, \dots$  be metric spaces. For  $x = (x_1, x_2, \dots)$  and  $y = (y_1, y_2, \dots)$ , with  $x_i \in X_i$  and  $y_i \in X_i$  we define

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n, y_n)}.$$

It is immediately verified that  $\rho$  is a metric on  $\prod_{n=1}^{\infty} X_n$ . It remains to be verified that the topology generated by this metric coincides with the Tikhonov topology. Let  $\Sigma_x$  be a Tikhonov neighborhood of the point  $x = (x_1, x_2, \dots)$ . By definition of the Tikhonov topology  $\Sigma_x$  contains a set of the form  $\left\{ y \in \prod_{n=1}^{\infty} X_n : \rho_{n_i}(x_{n_i}, y_{n_i}) < \varepsilon \right\}$ , where  $\varepsilon > 0$  and  $(n_1, \dots, n_k)$  is some set of indices. But this set in turn contains a ball (in the metric  $\rho$ ) of radius  $\frac{\varepsilon}{2^{\max(n_1, \dots, n_k) + 1}}$ .

Conversely let  $O_x$  be a "metric" neighborhood of the point  $x \in \prod_{n=1}^{\infty} X_n$ .

We may assume that  $O_x = \left\{ y \in \prod_{n=1}^{\infty} X_n : \rho(x, y) < \varepsilon \right\}$  for some  $\varepsilon > 0$ . Let  $N$  be such that  $\sum_{i=N+1}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{2}$ .

Consider the Tikhonov neighborhood of the point  $x$ :

$$\Sigma_x = \left\{ y \in \prod_{m=1}^{\infty} X_m : \rho_n(x_n, y_n) < \varepsilon/2, \quad n = 1, 2, \dots, N \right\}.$$

A simple computation shows that  $\Sigma_x \subset O_x$ . ■

THEOREM 5 (Tikhonov). *The product  $T = \prod_{\alpha \in \Omega} T_{\alpha}$  of compact topological spaces  $T_{\alpha}$  is compact.*

PROOF: Let  $\{\Sigma\}$  be an arbitrary centered system of subsets of  $X$ , where  $T = (X, \Sigma)$ . Consider a system  $\{\Sigma^m\}$  possessing the following properties:

- a)  $\{\Sigma\}$  is a subsystem of  $\{\Sigma^m\}$ ;
- b)  $\{\Sigma^m\}$  is centered;
- c) the system  $\{\Sigma^m\}$  is not a proper subsystem of any other centered system containing  $\{\Sigma\}$  as a subsystem.

It is obvious that we have introduced an order relation by inclusion, and the existence of the system  $\{\Sigma^m\}$  follows from Zorn's Lemma.



Consider a fixed element  $\Sigma^m$  of the system  $\{\Sigma^m\}$  and suppose that for any  $\alpha \in \Omega$  we set  $\Sigma_\alpha^m = \{f(\alpha) : f \in \Sigma^m\} \subset X_\alpha$ . We denote by  $\{\Sigma_\alpha^m\}$  the system of all the sets  $\Sigma_\alpha^m$  constructed above. The system  $\{\Sigma_\alpha^m\}$  is centered. Since the space  $T_\alpha$  is compact, there exists a point  $a_\alpha \in X_\alpha$  such that  $a_\alpha \in \bigcap_{\Sigma^m \in \{\Sigma^m\}} \overline{\Sigma}_\alpha^m$ . This follows from the assertion of Example 8 of Sec. 1.2.\* We shall show that the point  $a = \prod_{\alpha \in \Omega} a_\alpha$  belongs to the intersection  $\bigcap_{\Sigma^m \in \{\Sigma^m\}} \overline{\Sigma}^m$ . Then according to the same Example 8 the space  $T$  is compact.

Since the points  $a_{\alpha_0}$  belong to  $\bigcap_{\Sigma^m \in \{\Sigma^m\}} \overline{\Sigma}_{\alpha_0}^m$ , it follows that every open set  $G_{\alpha_0}$  of the space  $T_{\alpha_0}$  containing  $a_{\alpha_0}$  intersects each of the sets  $\Sigma_{\alpha_0}^m \in \{\Sigma_{\alpha_0}^m\}$ . Therefore the open set  $G^{(\alpha_0)} = \left\{x : x = \prod_{\alpha \in \Omega} x_\alpha, x_{\alpha_0} \in G_{\alpha_0}\right\}$  of the space  $T$  must intersect each of the sets  $\Sigma^m$  of the system  $\{\Sigma^m\}$ . According to property c) of the system  $\{\Sigma^m\}$  the set  $G^{(\alpha_0)}$  must belong to  $\{\Sigma^m\}$ . Again from property c) it follows that the intersection of any finite number of sets of type  $G^{(\alpha_0)}$ ,  $\alpha_0 \in \Omega$ , must also belong to the system  $\{\Sigma^m\}$ , and so such a set intersects each set  $\Sigma^m \in \{\Sigma^m\}$ . Every open set of the space  $T$  containing the point  $a$  by definition contains some intersection of this type. Consequently the point  $a = \prod_{\alpha \in \Omega} a_\alpha$  belongs to  $\bigcap_{\Sigma^m \in \{\Sigma^m\}} \overline{\Sigma}^m$ . ■

## 5.6. The Stone-Weierstrass Theorem

The classical theorem of Weierstrass, which asserts that the set of polynomials is dense in  $C[a, b]$ , is known from analysis. We shall prove here a great generalization of it obtained by Stone. In the proof we shall need the following corollary of the Weierstrass theorem: for any  $a > 0$  and  $\varepsilon > 0$  there exists a polynomial  $p(x)$  such that  $\sup_{x \in [-a, a]} ||x| - p(x)| < \varepsilon$ .

**THEOREM (Stone-Weierstrass).** *Let  $T = (X, \Sigma)$  be a compact Hausdorff space and  $C(X)$  the space of all real-valued continuous functions on  $X$ . Let  $B(X)$  be a subset of  $C(X)$  such that:*

- a)  $f, g \in B(X)$  implies that  $f \cdot g$  and  $\alpha f + \beta g$  belong to  $B(X)$  for any real numbers  $\alpha$  and  $\beta$ ;
- b) constant functions belong to  $B(X)$ ;

\*We remark that the proof of the assertion of this example given at the end of Sec. 1.2 for the case of topological spaces is the same as for the case of a metric space.

c) the limit of every uniformly convergent sequence  $\{f_n\}$  of functions of  $B(X)$  belongs to  $B(X)$ .

Then  $B(X) = C(X)$  if and only if for any two distinct points  $x_1$  and  $x_2$  of the set  $X$  there exists a function  $f \in B(X)$  such that  $f(x_1) \neq f(x_2)$ .

PROOF: Indeed, according to Theorem 2 a compact Hausdorff space is normal, and by Uryson's Lemma for any points  $x_1$  and  $x_2$  with  $x_1 \neq x_2$  there exists a function  $f \in C(X)$  such that  $f(x_1) \neq f(x_2)$ .

Conversely let  $f \vee g = \max(f(x), g(x))$  and  $f \wedge g = \min(f(x), g(x))$ . It is easy to see that

$$f \vee g = \frac{f+g}{2} + \frac{|f-g|}{2}, \quad f \wedge g = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

Since the space is compact, every function  $f \in C(X)$  is bounded and so by Weierstrass' theorem  $|f(x)|$  is the limit of a uniformly convergent sequence of polynomials in  $f$ , i.e.,  $|f| \in B(X)$  if  $f \in B(X)$ . Consequently the set  $B(X)$  is closed under the operations  $\vee$  and  $\wedge$ .

Let  $g(x) \in C(X)$  and let  $x_1$  and  $x_2$  be arbitrary points of  $X$  with  $x_1 \neq x_2$ . Choose a function  $h \in B$  such that  $h(x_1) \neq h(x_2)$ . Form a linear combination  $f_{x_1, x_2} = \alpha h + \beta$  with  $\alpha, \beta \in \mathbf{R}$  unknown for the time being. We shall determine the numbers  $\alpha$  and  $\beta$  by the system

$$\begin{aligned} \alpha h(x_1) + \beta &= g(x_1), \\ \alpha h(x_2) + \beta &= g(x_2). \end{aligned}$$

Now let  $\varepsilon > 0$  be arbitrary and  $y \in X$ . For any point  $x \in X$  there exists a neighborhood  $\Sigma_x$  such that  $f_{x,y}(u) > g(u) - \varepsilon$  for any point  $u \in \Sigma_x$ . Let the sets  $\Sigma_{x_1}, \Sigma_{x_2}, \dots, \Sigma_{x_n}$  cover the compact space  $T$ ; set  $f_y = f_{x_1,y} \vee \dots \vee f_{x_n,y}$ . Then  $f_y \in B(X)$  and  $f_y(u) > g(u) - \varepsilon$  for any point  $u \in X$ . Moreover  $f_y(y) = g(y)$ , so that  $f_{x,y}(y) = g(y)$ . Consequently there exists a neighborhood  $\Sigma_y$  of the point  $y$  such that  $f_y(u) < g(u) + \varepsilon$  for any point  $u \in \Sigma_y$ . Let the sets  $\Sigma_{y_1}, \dots, \Sigma_{y_k}$  cover  $T$ ; we set  $f = f_{y_1} \wedge \dots \wedge f_{y_k}$ . Then  $f \in B(X)$  and  $f(u) > g(u) - \varepsilon$  for any point  $u \in X$ , since  $f_{y_j}(u) > g(u) - \varepsilon$  for any  $u \in X$ . Moreover for any point  $u \in X$  and in particular for  $u \in \Sigma_{y_j}$  the inequalities  $f(u) \leq f_{y_j}(u) < g(u) + \varepsilon$ . Consequently  $|f(u) - g(u)| < \varepsilon$  for any point  $u \in X$ , which was to be proved. ■

We remark that in the case of a closed interval of the real line the polynomials obviously possess the properties of the subset  $B(X)$  shown in the theorem. In this case we obtain the classical Weierstrass theorem on approximation of a continuous function on a closed interval by a uniformly convergent sequence of polynomials.



The Stone-Weierstrass Theorem also has an interesting corollary: let  $T_\alpha = (X_\alpha, \sigma_\alpha)$ ,  $\alpha \in A$ , be a family of compact Hausdorff spaces. Let  $T = (X, \Sigma) = \prod_{\alpha \in A} T_\alpha$ . In the space  $C(X)$  consider the set of functions depending only on a finite number of the coordinates (i.e., of the form  $f(x) = f(x_{n_1}, \dots, x_{n_k})$ ). It follows from the Stone-Weierstrass Theorem that the set of such functions is dense in  $C(X)$ . Hence we can deduce that any function of  $C(X)$  depends only on a countable number of coordinates (the set of indices  $A$  may be arbitrary here). In this reasoning we have made implicit use of Tikhonov's Theorem that the product of compact spaces is compact.

#### EXAMPLES

1. The rectangle  $-\infty < a_i \leq x_i \leq b_i < +\infty$ ,  $a_i < b_i$ ,  $i = 1, \dots, n$  in  $\mathbf{R}^n$  is compact.

This follows from the Tikhonov Theorem and the fact that a closed interval on the real line is compact in the topology induced by the metric  $\rho(x, y) = |x - y|$ .

2. The product of two Hausdorff spaces, two regular spaces, or two completely regular spaces is Hausdorff, regular, or completely regular respectively.

#### EXERCISES

1. Give an example of a normal nonmetrizable topological space.
2. Let  $\Sigma_1$  and  $\Sigma_2$  be open subsets of a normal space, and let  $F$  be a closed set contained in the union  $\Sigma_1 \cup \Sigma_2$ . Show that  $F = F_1 \cup F_2$ , where  $F_i$  is a closed subset of  $\Sigma_i$ ,  $i = 1, 2$ .
3. Let  $T_x = \mathbf{R}^1$  for  $x \in \mathbf{R}^1$ , and let  $T = \prod_{x \in \mathbf{R}^1} T_x$ . Then  $T$  contains a countable subset whose closure coincides with  $T$ , but there is no countable basis for its topology.
4. A space is called *completely normal* if each open subset of it is normal. Give an example of a topological space that is not completely normal.
5. Prove that a metric space is complete normal.
6. Suppose we require that in a topological space satisfying Axiom  $T_1$  any two disjoint closed subsets have neighborhoods whose closures are disjoint. Will the resulting class of topological spaces be smaller than the class of normal spaces?
7. Prove that on any infinite set there exists a topology satisfying the Hausdorff axiom and such that no point of the set is isolated with respect to the topology.

8. Give an example of a topological space in which all one-point sets are closed (i.e., Axiom  $T_1$  holds) and yet any two non-empty open sets intersect.
9. Let  $T$  be a Hausdorff space such that the set of isolated points is finite. Prove that the space  $T$  is normal.
10. Prove that a regular topological space whose topology has a countable basis is metrizable. In particular a compact Hausdorff space is metrizable if and only if there is a countable basis for its topology.



