

Chapter 2

Vector Spaces

1. TOPOLOGICAL VECTOR SPACES

In the first chapter we introduced the important classes of metric and topological spaces and studied their basic properties. The necessity often arises, however, of considering sets on which two operations are defined, addition and scalar multiplication, having certain properties. Such sets, called *vector spaces*, are the main object of study in the present chapter. The most interesting results are obtained in the case when a vector space is endowed with the further structure of a topological space or some other structure, for example, a norm which defines the topology in the space and is such that the linear operations are continuous in the topology. Topological vector spaces, the so-called "*F*-spaces," and normed spaces play an exceedingly important role in analysis and therefore will be studied in detail. It is customary to distinguish several important principles in the theory of such spaces. These principles are the uniform boundedness principle, the open mapping principle, and the principle of extension of linear functionals. This material forms the subject matter of the present chapter.

1.1. Group, Ring, Field, and Vector Space

DEFINITION 1. A set G of elements of arbitrary nature is called a *group* if there is an operation in G associating with each pair of elements x and y of G a certain element $w = xy$, called the *product* of the elements x and y , and satisfying the following axioms.

- 1) $(xy)z = x(yz)$ (associativity);

2) G has a left identity, i.e., an element e such that $ex = x$ for any $x \in G$;

3) for each element $x \in G$ there exists a left inverse, i.e., an element x^{-1} such that $x^{-1}x = e$.

If in addition for any two elements x and y the equality $xy = yx$ holds, the group is called *commutative* or *abelian*. For commutative groups the group operation is usually written as addition, $w = x + y$, the identity is called zero and denoted 0 , and the element x^{-1} inverse to x is called the negative of x and denoted $-x$. Such groups are called *additive groups*.

If U and V are two subsets of a group G , we denote by UV the subset consisting of all elements of the form xy , where $x \in U$ and $y \in V$. We denote by U^{-1} the subset consisting of all elements of the form x^{-1} , where $x \in U$.

DEFINITION 2. A set G_T is called a *topological group* if:

1) G_T is a group;

2) G_T is a topological space;

3) the group operations in G_T are continuous on the topological space G_T , i.e., if x and y are two elements of G_T , then for every neighborhood W of the element $w = xy$ there exist neighborhoods U and V of the elements x and y respectively such that $UV \subset W$, and for every neighborhood Σ of the element x^{-1} there exists a neighborhood U of the element x such that $U^{-1} \subset \Sigma$.

EXAMPLES

1. It is easy to see that a left identity e of the group G is also a right identity, i.e., $xe = x$ for any $x \in G$.

Indeed $x^{-1}xx^{-1} = x^{-1}$. Multiplying this equality on the left by a left inverse of the element x^{-1} , we obtain $xx^{-1} = e$, i.e., a left inverse is also a right inverse. Moreover, x is the element inverse to x^{-1} . We then have $xe = xx^{-1}x = ex = x$, i.e., the left identity is also a right identity.

2. In a group G each of the equations

$$ax = b,$$

$$ya = b$$

in the unknowns x and y has a unique solution.

Indeed, it is easy to verify by substitution that $a^{-1}b$ is a solution of the first equation and ba^{-1} is a solution of the second. These solutions are unique, for if we multiply the first equation on the left by a^{-1} , we obtain

$x = a^{-1}b$, and in exactly the same way we obtain $y = ba^{-1}$ from the second equation.

The concept of a ring is an important one. We give the following definition.

DEFINITION 3. A *ring* K is a set on which two operations called addition and multiplication are defined having the following properties.

- 1) K is an abelian group with respect to addition;
- 2) $(xy)z = x(yz)$;
- 3) $x(y+z) = xy + xz$, $(y+z)x = yx + zx$.

A ring is called *commutative* if the equality $xy = yx$ holds in it for all $x, y \in K$. If the ring K contains an element 1 such that $1 \cdot x = x \cdot 1 = x$ for all $x \in K$, we shall say that K is a *ring with identity*.

DEFINITION 4. A *field* P is a commutative ring with identity whose nonzero elements form a group with respect to multiplication.

Throughout our investigations we shall assume that P is the field of real or complex numbers.

We now turn to the definition of the most important concept, that of a vector space.

DEFINITION 5. A *vector space* V over a field P is a set in which two operations are defined called vector addition and scalar multiplication, possessing the following properties:

- 1) V is an abelian group with respect to vector addition;
- 2) the following relations hold: $\alpha(x+y) = \alpha x + \alpha y$, $(\alpha+\beta)x = \alpha x + \beta x$, $\alpha(\beta x) = (\alpha\beta)x$, $1 \cdot x = x$, where $1, \alpha, \beta \in P$ and $x, y \in V$.

The elements $x, y, \dots \in V$ are called *vectors* in V and the elements of the field P , i.e., $1, \alpha, \beta, \dots$, are called *scalars*.

Let us clarify 1) and 2) in more detail. The fact that V is an additive abelian group means that the sum $x+y$ of two elements x and y of V is defined and is an element of the same set, and that the addition operation satisfies the conditions

- a) $x+y = y+x$ (commutativity);
- b) $x+(y+z) = (x+y)+z$ (associativity);
- c) there exists a uniquely determined element 0 such that

$$0 + x = x$$

for any $x \in V$;

d) for each element $x \in V$ there exists a uniquely determined element $(-x)$ of the same space such that

$$(-x) + x = 0.$$

From now on we shall write $x - y$ instead of $(-y) + x$.

As mentioned, the element 0 is called zero, and the element $(-x)$ is called the negative of x .

Property 2) means that multiplication of the elements x, y, z, \dots of V by scalars $\alpha, \beta, \lambda, \mu, \dots$ from the field P is defined, that the element λx , for example, again belongs to V , and that the four relations in 2) hold.

As simple corollaries of the axioms of 1) and 2) of the definition of a vector space we obtain the following propositions.

1. $0x = 0$ (We remark that the symbol 0 denotes both the number 0 and the zero element of the vector space. It will be clear from the context which is meant in any given case.)

Indeed, $x = 1 \cdot x = (0 + 1)x = 0 \cdot x + 1 \cdot x = 0 \cdot x + x$.

Hence (using the equality $x = 0 \cdot x + x$) we have $(-x) + x = (-x) + 0 \cdot x + x$,
or

$$0 = 0 + 0 \cdot x = 0 \cdot x.$$

2. $(-1)x = -x$, since

$$(-1)x + x = (-1 + 1)x = 0 \cdot x = 0,$$

and consequently x is the negative of $(-1) \cdot x$.

3. $\lambda \cdot 0 = 0$, i.e., any scalar λ multiplied by the zero vector in V is the zero vector in V . Indeed

$$\lambda \cdot 0 = \lambda[(-x) + x] = \lambda(-x) + \lambda x = (-\lambda)x + \lambda x = \lambda x - \lambda x = 0.$$

4. Let $\lambda x = \mu x$ and $x \neq 0$. Then $\lambda = \mu$.

Indeed if $\lambda x = \mu x$, then $\lambda x - \mu x = 0$ or $(\lambda - \mu)x = 0$. If now $\lambda \neq \mu$, then $x = \frac{1}{\lambda - \mu}(\lambda - \mu)x = \frac{1}{\lambda - \mu} \cdot 0 = 0$, contradicting the hypothesis that $x \neq 0$.

5. We note finally the following interesting fact: if V is a vector space the commutativity of addition is a consequence of the other axioms. Indeed

$$\begin{aligned} -(x + y) + (y + x) &= (-1)(x + y) + (y + x) = (-1)x + (-1)y + y + x \\ &= (-1)x + [(-1)y + y] + x = (-1)x + 0 + x = (-1)x + x = 0. \end{aligned}$$

A vector of the form $\sum_{i=1}^n \alpha_i x_i$, where $\alpha_i \in P$, $x_i \in V$, is called a *linear combination* of the vectors x_i , $i = 1, 2, \dots, n$.

If V is a vector space, $A \subset V$, $B \subset V$, $x \in V$, and $\alpha \in P$, then we shall use the notation

$$x - A = \{x - a : a \in A\}, A + B = \{a + b : a \in A, b \in B\}, \alpha A = \{\alpha a : a \in A\}.$$

DEFINITION 6. A set of elements $G \subset V$ is called a *linear manifold* if it contains all linear combinations of elements that belong to it. We shall say that a linear manifold is *spanned* by the set A if it coincides with the set of all linear combinations of elements of A .

DEFINITION 7. A set of vectors x_1, x_2, \dots, x_n of a vector space is called *linearly independent* if the equality $\sum_{i=1}^n \alpha_i x_i = 0$ implies that all $\alpha_i = 0$. An infinite set of vectors $\{x_\alpha\}$ in a vector space is called linearly independent if any finite subset of it is linearly independent.

DEFINITION 8. A set $B \subset V$ is called an *algebraic basis* if any vector x of the space can be written uniquely as a linear combination $x = \sum_{i=1}^n \alpha_i x_i$, where $\alpha_i \in P$ and $x_i \in B$.

The cardinality of a set of elements constituting a basis is called the *dimension* of the vector space V .

We shall illustrate the definitions just given for a linear manifold and linear independence with some examples. Suppose, for example, V is some vector space and $x \in V$, $x \neq 0$. The set $\{\lambda x\}$, $\lambda \in P$, obviously forms a linear manifold. The dimension of this linear manifold, regarded as a vector space, is 1.

If $C[a, b]$ is the space of continuous functions and $\{P_n\}$ the set of all polynomials on the closed interval $[a, b]$, then $\{P_n\}$ is an infinite-dimensional linear manifold.

Let V be a vector space and V^1 some linear manifold, $V^1 \subset V$. We shall say that two elements x and y of V are *equivalent* ($x \sim y$) if $x - y \in V^1$. This relation has the properties of reflexivity, symmetry, and transitivity (i.e., $x \sim x$; if $x \sim y$, then $y \sim x$; if $x \sim y$ and $y \sim z$, then $x \sim z$, respectively). Therefore it is an equivalence relation and partitions V into disjoint classes. An equivalence class is called a *coset* over the linear manifold V^1 . The set of all cosets over V^1 is called the *quotient space* of V over V^1 and denoted V/V^1 . Linear operations are introduced on the quotient space according to the following rule: if ξ and η are two elements of V/V^1 , their

sum is the element $\zeta = \xi + \eta$ containing all the elements $x + y$, where x is a representative of the class ξ and y a representative of the class η . Similarly we define the product of an element $\xi \in V/V^1$ by a scalar α . It is not difficult to verify that if V is a space of dimension n and the dimension of V^1 is k , then $k \leq n$ and the dimension of V/V^1 is $l = n - k$. This dimension is called the *codimension* of the linear manifold V^1 in V .

If V^1 has finite codimension l , then one can choose elements in V , say x_1, x_2, \dots, x_l , such that every element $x \in V$ can be uniquely represented in the form $x = \sum_{i=1}^l \alpha_i x_i + y$, where $\alpha_i \in P$, $i = 1, \dots, l$, and $y \in V^1$. (Recall that P is the field of real or complex numbers.)

A vector space V is called the *algebraic direct sum* of the vector spaces V_1 and V_2 if V_1 and V_2 are linear manifolds in V and any element $x \in V$ is uniquely representable in the form $x = x_1 + x_2$, where $x_1 \in V_1$ and $x_2 \in V_2$.

If V_1 and V_2 are vector spaces over the field P , then the Cartesian product

$$V = V_1 \times V_2$$

(cf. Sec. 1.1.1) becomes a vector space if we define the operations in it by

$$\begin{aligned}(x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2), \\ \lambda(x_1, x_2) &= (\lambda x_1, \lambda x_2),\end{aligned}$$

where $x_1 \in V_1$, $x_2 \in V_2$, so that $(x_1, x_2) \in V_1 \times V_2$.

EXAMPLES

1. Any two bases of a vector space have the same cardinality. When there exists a finite basis, this fact is well known from linear algebra.

2. If K is a ring, a set $K_1 \subset K$ is called a *subring* if the elements of K_1 form a ring with respect to the operations in K . We call a subring $I \subset K$ a *right ideal* if the following axioms hold:

- 1) $Ix \subset I$ for all $x \in K$, where $Ix = \{y \cdot x\}$, $y \in I$;
- 2) $0 \neq I \neq K$.*

The definition of a left ideal is similar. A subring $I \subset K$ that is both a left ideal and a right ideal is called a *two-sided ideal*.

An ideal (right, left, or two-sided) is called a *maximal ideal* if it is not contained in any other ideal (right, left, or two-sided respectively).

*Such a right ideal is sometimes called a *proper ideal*.

3) Let P be a field. A set A is called an *algebra over the field P* if A is both a ring and a vector space over P and

$$\alpha(xy) = (\alpha x)y = x(\alpha y)$$

for all elements x and y in A and all scalars α .

4) We give examples of the commonest rings. Let $C(X)$ be the set of all complex functions $\{x(t)\}$ defined and continuous on a topological space X . Obviously $C(X)$ is a commutative ring with the usual addition and multiplication of functions. In exactly the same way $C^k(X)$, the set of k times continuously differentiable functions is a commutative ring with the usual addition and multiplication operations.

Let V be the space of polynomials with complex coefficients whose degree is at most n , and let K be the set of mappings of V into V of the form

$$f \rightarrow \sum_{k=0}^m a_k D^k(f),$$

where $f \in V$, $a_k \in P$, and D is the differentiation operator. Then K is a commutative ring and $D^{n+1} = 0$.

The set of square matrices of order n , where $n > 1$, with the usual matrix operations forms a noncommutative ring.

5. As an example of a two-sided ideal in the ring $C[a, b]$, where $[a, b]$ is a closed interval of the real line (cf. Example 4, the ring $C(X)$), one can take the set of all functions of $C[a, b]$ that vanish on the closed interval $[0, 1/2]$. It can be verified that a maximal two-sided ideal of this ring is the set of functions of $C[a, b]$ vanishing at some given point of the interval $[a, b]$.

1.2. Linear Transformations. The Space of Transformations

DEFINITION 9. A mapping A of a vector space V_1 into a second vector space V_2 over the same field as V_1 is called a *linear transformation* (denoted $A : V_1 \rightarrow V_2$) if the following axioms hold:

- 1) $A(x + y) = Ax + Ay$ for any x and y in V_1 ;
- 2) $A(\alpha x) = \alpha Ax$ for any $x \in V_1$ and any $\alpha \in P$.

If $V_1 = V_2 = V$, A is called a *linear operator on V* .

We introduce the notion of the *sum* of two transformations $A : V_1 \rightarrow V_2$ and $B : V_1 \rightarrow V_2$ as follows: $(A + B)x = Ax + Bx$ for $x \in V_1$. It is obvious that $(A + B)$ is a linear transformation from V_1 into V_2 , so that $(A + B) : V_1 \rightarrow V_2$. In exactly the same way we define the *product* of the linear transformation A by a scalar $\alpha \in P$, namely $(\alpha A)x = \alpha(Ax)$ for any

$x \in V_1$. Finally if $A : V_1 \rightarrow V_2$ and $B : V_2 \rightarrow V_3$, we define the *composition* of the linear transformations B and A by the rule $(BA)x = B(Ax)$. The composition is a linear transformation from V_1 to V_3 .

In this way we arrive at an important concept—the space of linear transformations.

DEFINITION 10. The set of linear transformations from V_1 into V_2 forms a vector space with the operations of addition of the transformations A and B and multiplication of the transformation A by the scalar $\alpha \in P$ introduced above. This space is called the *space of linear transformations* and is denoted $(V_1 \rightarrow V_2)$.

If $V_1 = V_2 = V$, then $(V \rightarrow V)$ is a ring.

Thus we have constructed a new space $(V_1 \rightarrow V_2)$ whose elements are linear transformations.

We now consider the special case of a linear transformation A from the vector space V_1 into the field of real or complex numbers. In this connection we give the following definition.

DEFINITION 11. A linear transformation $A : V_1 \rightarrow V_2$ is called a *linear functional* if $V_2 \subset P$, where P is the field of scalars.

Thus a functional maps a vector space into its scalar field, i.e., a functional is a numerical-valued function.

We recall that throughout this book the field P over which the vector space V is defined coincides with either \mathbf{R}^1 or the field \mathbf{C} of complex numbers. (In the first case V is called a *real* vector space, in the second, a *complex* vector space.)

1.3. Banach Spaces

Normed and Banach spaces play an important role in functional analysis.

DEFINITION 12. A vector space N over the field P of real or complex numbers is said to be *normed* if to each $x \in N$ there is assigned a nonnegative real number $\|x\|$, called the *norm* of the element and satisfying the following axioms:

- 1) $\|x + y\| \leq \|x\| + \|y\|$ for all x and y in N ;
- 2) $\|\alpha x\| = |\alpha| \|x\|$ for any $x \in N$ and $\alpha \in P$;
- 3) $\|x\| > 0$ if $x \neq 0$, and $\|0\| = 0$, where 0 is the zero element of N .

Each normed space can be regarded as a metric space in which the function $\rho(x, y)$ is defined as $\rho(x, y) = \|x - y\|$.

Obviously from axioms 1)–3) of Definition 12 we see that the function $\rho(x, y)$ satisfies axioms 1)–3) in the definition of distance (cf. Sec. 1.2.1).

Thus normed spaces have all the properties of metric spaces. In particular all the concepts we introduced in Secs. 1.2 and 1.3 carry over to normed spaces, and normed spaces are topological spaces.

We now turn to the definition of a Banach space.

DEFINITION 13. A *Banach space* B is a normed space that is complete in the metric $\rho(x, y) = \|x - y\|$ defined by its norm.

Examples of Banach spaces are the metric spaces in the examples of Sec. 1.2.1 such as \mathbf{R}^n , $C[a, b]$, $C^n[a, b]$ ($n \geq 1$), l^2 , l^p , and m , in all of which the norm of the element x is defined to be the distance from x to 0, i.e., $\|x\| = \rho(x, 0)$.

We now give an example to show that the geometry in a Banach space may present features that run counter to intuition. We define, for example, for a vector $f = (f_1, f_2)$ of the two-dimensional plane the norm $\|f\| = |f_1| + |f_2|$.

The unit ball in the normed space \mathbf{R}_1^2 thus introduced is a square of side $\sqrt{2}$ with vertices at the points $(1, 0)$, $(0, -1)$, $(-1, 0)$, and $(0, 1)$ on the OX - and OY -axes.

We now draw a line passing through the origin at an angle of 45° to the OX -axis. We denote this line by M and regard M as a subspace of the normed space \mathbf{R}_1^2 . Any vector g belonging to M has coordinates (g_1, g_1) , i.e., $g = (g_1, g_1)$. Therefore if we consider the vector $f = (1, 0)$ and write $\|f - g\| = |1 - g_1| + |g_1|$, the norm of the difference of the vectors f and g attains its minimum (equal to 1) at any g_1 such that $0 \leq g_1 \leq 1$.

Thus the minimum distance from the vector f to the subspace M is attained on an infinite set of vectors g of the subspace. It is clear that if we had considered the plane \mathbf{R}^2 with the usual Euclidean distance, this minimum would have been attained at only one vector.

In some normed spaces, when M is infinite-dimensional, this minimum may fail to be attained at all.

A Banach space is said to be *uniformly convex* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\frac{1}{2}\|f + g\| > 1 - \delta$ and $\|f\| = \|g\| = 1$, then $\|f - g\| < \varepsilon$.

If a Banach space is uniformly convex, then the vector g of the subspace M giving the minimum of $\|f - g\|$ for some f of the space is unique if it exists at all.

1.4. Convex Sets, The Minkowski Functional, Seminorms

The concept of a convex set in a vector space plays an important role in functional analysis. In this connection we give the following definition.

DEFINITION 14. A subset B of a vector space V is called *convex* if for any elements x and y of B and any scalar t with $0 \leq t \leq 1$ the relation $tx + (1-t)y \in B$ holds.

In other words it is required that the set B contain the line segment joining any two of its points.

The property of convexity for a set B can also be written as follows:

$$tB + (1-t)B \subset B, \quad 0 \leq t \leq 1.$$

The following simple property holds.

PROPERTY 1. *The intersection of any number of convex sets is a convex set.*

PROOF: Indeed, let $C = \bigcap_{\alpha} C_{\alpha}$, where each C_{α} is convex. Let x and y be arbitrary points of C . Then x and y belong to each C_{α} . By virtue of the convexity of each C_{α} the line segment joining the points x and y is contained in each of these sets. But then this line segment is contained in the intersection of all C_{α} , i.e., it is contained in the set C . Hence the set C is convex. ■

We shall also use the concept of a balanced set.

DEFINITION 15. A set B contained in a vector space V is called *balanced* if $\alpha B \subset B$ for any scalar $\alpha \in P$ such that $|\alpha| \leq 1$, i.e., for any element $x \in B$ the element αx belongs to B if $|\alpha| \leq 1$.

For example, a disk in the plane and a ball in \mathbf{R}^n with center at the origin are convex and balanced sets. A rectangle in \mathbf{R}^n , i.e., a set whose elements x satisfy $a_i \leq x_i \leq b_i$, $i = 1, 2, \dots, n$ is a convex set but in general not a balanced set.

We now introduce another concept.

DEFINITION 16. A set $B \subset V$ is called *absolutely convex* if $\lambda x + \mu y \in B$ for any x and y in B and any λ and μ in P with $|\lambda| + |\mu| \leq 1$.

The properties enumerated below can be established using the definitions given above.

PROPERTY 2. *A set B is absolutely convex if and only if it is convex and balanced.*

PROOF: Let B be absolutely convex. Then it is obviously convex and balanced.

Conversely suppose B is convex and balanced. Let $x, y \in B$ and $|\lambda| + |\mu| \leq 1$. If $\lambda = 0$ (or $\mu = 0$), obviously $\lambda x + \mu y = \mu y \in B$ (or $\lambda x + \mu y = \lambda x \in B$) since B is balanced. Now let $\lambda \neq 0$ and $\mu \neq 0$. Then

$$x_1 = \frac{\lambda}{|\lambda|}x \in B, \quad y_1 = \frac{\mu}{|\mu|}y \in B,$$

again because B is balanced.

We now write the relation

$$\begin{aligned} \lambda x + \mu y &= (|\lambda| + |\mu|) \left(\frac{|\lambda|}{|\lambda| + |\mu|} \cdot \frac{\lambda x}{|\lambda|} + \frac{|\mu|}{|\lambda| + |\mu|} \cdot \frac{\mu y}{|\mu|} \right) \\ &= \alpha(tx_1 + (1-t)y_1), \end{aligned}$$

where

$$\begin{aligned} 0 < \alpha = |\lambda| + |\mu| \leq 1, \quad 0 \leq t = \frac{|\lambda|}{|\lambda| + |\mu|} \leq 1, \\ x_1 = \frac{\lambda}{|\lambda|}x \in B, \quad y_1 = \frac{\mu}{|\mu|}y \in B. \end{aligned}$$

Consequently $\lambda x + \mu y = \alpha(tx_1 + (1-t)y_1) \in B$, since by the convexity of B the element $z = tx_1 + (1-t)y_1 \in B$, and by the fact that B is balanced $\alpha z \in B$ for $0 < \alpha \leq 1$. ■

PROPERTY 3. *Let B_1 and B_2 be convex [resp. absolutely convex] subsets of V and $\lambda \in P$. Then the sets $B_1 + B_2$ and λB_1 are convex [resp. absolutely convex].*

PROOF: We shall prove, for example, that $B_1 + B_2$ is convex. Let $x, y \in B_1 + B_2$, and $0 \leq t \leq 1$. Then $x = x_1 + x_2$, and $y = y_1 + y_2$, where $x_1, y_1 \in B_1$ and $x_2, y_2 \in B_2$. We have

$$\begin{aligned} tx + (1-t)y &= t(x_1 + x_2) + (1-t)(y_1 + y_2) \\ &= tx_1 + (1-t)y_1 + tx_2 + (1-t)y_2 \in B_1 + B_2, \end{aligned}$$

since by the convexity of B_1 and B_2 the element $tx_1 + (1-t)y_1$ belongs to B_1 and the element $tx_2 + (1-t)y_2$ belongs to B_2 . The other assertions of Property 3 are established in exactly the same way. ■

PROPERTY 4. *Let B be a nonempty absolutely convex set in V . Then $0 \in B$ and $\lambda B \subset \mu B$ if $|\lambda| \leq |\mu|$.*

PROOF: It is obvious that $0 \in B$. Now let $|\lambda| \leq |\mu| \neq 0$ and let $x \in \lambda B$. Then $x = \lambda x_1$, where $x_1 \in B$, and by virtue of the absolute convexity of B we have $\frac{\lambda}{\mu} x_1 \in B$ (since $|\lambda|/|\mu| \leq 1$). In other words $\frac{\lambda}{\mu} x_1 = y \in B$, i.e., $\lambda x_1 = \mu y$, or $\lambda B \subset \mu B$.

The case when $\lambda = \mu = 0$ is obvious. ■

If B is an arbitrary nonempty subset of V then the set of all finite linear combinations $\sum \lambda_i x_i$, where $\lambda_i \geq 0$, $\sum \lambda_i = 1$, $x_i \in B$, is called the *convex hull* of the set B . It is obvious that the convex hull of B is the smallest convex set containing B .

The set of all linear combinations $\sum \lambda_i x_i$, where $\sum |\lambda_i| \leq 1$, $x_i \in B$, is called the *absolute convex hull* of the set B . In exactly the same way the absolute convex hull of a set B is the smallest absolutely convex set containing B .

DEFINITION 17. A set B situated in a vector space V is called *absorbing* if for each $x \in V$ there exists a number $\lambda = \lambda(x) > 0$ such that $x \in \mu B$ for all μ such that $|\mu| \geq \lambda$.

Geometrically this property means that on every ray emanating from zero there is an interval starting at zero and contained in B .

For example, any neighborhood of zero in \mathbb{R}^n is an absorbing set.

By Property 4 if a set B is absolutely convex it will be absorbing if and only if for any $x \in V$ there exists $\lambda > 0$ such that $x \in \lambda B$, i.e., $V = \bigcup_{\lambda > 0} \lambda B$

(or $V = \bigcup_{n=1}^{\infty} nB$).

We now turn to the important concept of a seminorm and to the study of the properties of the seminorm of a Minkowski functional.

Let $p(x)$ be a real-valued functional (real-valued function) defined on a vector space V .

The functional is called *subadditive* if 1) $p(x_1 + x_2) \leq p(x_1) + p(x_2)$ for any x_1 and x_2 in V , *positive-homogeneous* if 2) $p(\lambda x) = \lambda p(x)$ for $\lambda \geq 0$, and *homogeneous* if 3) $p(\lambda x) = |\lambda| p(x)$ for any λ .

DEFINITION 18. A real-valued functional possessing properties 1) and 3) is called a *seminorm*.

A functional satisfying properties 1) and 2) is called a *gauge function*.

We note the following:

- a) $p(0) = 0$ for any gauge function or seminorm.
- b) If p is a seminorm, then $p(x) \geq 0$ for any $x \in V$.

Indeed,

$$0 = p(0) = p(x + (-x)) \leq p(x) + p(-x) = 2p(x),$$

i.e., $p(x) \geq 0$.

c) For a seminorm p we have the relation

$$|p(x) - p(y)| \leq p(x - y).$$

Indeed, by the subadditivity we have

$$p(x) = p(x - y + y) \leq p(x - y) + p(y).$$

Interchanging x and y and remarking that

$$p(x - y) = p((-1)(y - x)) = p(y - x),$$

we obtain

$$p(y) = p(y - x + x) \leq p(x - y) + p(x).$$

The required relation follows from these two inequalities.

d) For a gauge function p we have the analogous relation

$$|p(x) - p(y)| \leq \max(p(x - y), p(y - x)).$$

Here we remark that the estimate takes this form because of the absence of the equality $p(x - y) = p(y - x)$.

We remark finally that although $p(0) = 0$ for any seminorm, it does not in general follow that $x = 0$ when $p(x) = 0$. A seminorm for which the relation $p(x) = 0$ implies $x = 0$ is obviously a norm.

DEFINITION 19. Let B be an absorbing set in a vector space V . The *Minkowski functional* of the set B is defined by the formula

$$p_B(x) = \inf_{\mu} \{\mu : \mu > 0, \quad x \in \mu B\},$$

where $x \in V$.

We remark that $p_B(x) < \infty$ for all $x \in V$, since the set B is absorbing.

It turns out that a seminorms on V are precisely the Minkowski functionals of all balanced convex absorbing sets.

Since the property of being balanced and convex is equivalent to being absolutely convex, it thereby results that the seminorms on V are the

Minkowski functionals of all absolutely convex absorbing sets. To be specific, we have the following property.

PROPERTY 5. *If p is a nonnegative gauge function, then for any $\lambda > 0$ the sets $\{x : p(x) < \lambda\}$ and $\{x : p(x) \leq \lambda\}$ are convex and absorbing. If p is a seminorm, these sets are also balanced, i.e., absolutely convex.*

Conversely to each convex absorbing set $B \subset V$ there corresponds a Minkowski functional $p_B(x) = \inf_{\mu} \{\mu : \mu > 0, x \in \mu B\}$ with

$$\{x : p_B(x) < 1\} \subset B \subset \{x : p_B(x) \leq 1\}.$$

If in addition B is balanced, then p_B is a seminorm.

PROOF: We shall prove that the set $\{x : p(x) < \lambda\}$, $\lambda > 0$, is convex. Let x and y be two arbitrary points of this set and $0 \leq t \leq 1$. Then by the subadditivity and positive homogeneity of p the point $tx + (1-t)y$ also belongs to this set:

$$p(tx + (1-t)y) \leq p(tx) + p((1-t)y) = tp(x) + (1-t)p(y) < \lambda.$$

We shall show that the set $\{x : p(x) < \lambda\}$, $\lambda > 0$, is absorbing. Let $\max(p(x), p(-x)) = a$. Choose μ satisfying the inequality $|\mu| \geq (a + \varepsilon)/\lambda$, $\varepsilon > 0$. Then*

$$p\left(\frac{x}{\mu}\right) = \frac{1}{|\mu|} p(\operatorname{sgn} \mu \cdot x) \leq \frac{\lambda}{a + \varepsilon} p(\operatorname{sgn} \mu \cdot x) < \lambda,$$

from which it follows that $x \in \mu\{x : p(x) < \lambda\}$ for $\lambda > 0$. Consequently the set $\{x : p(x) < \lambda\}$ is absorbing.

We now prove the second half of the proposition. Since B is absorbing, we have $p_B(x) < +\infty$. Now let $\lambda > 0$. We shall verify** that the functional p_B is positive-homogeneous. Since $\lambda x \in \mu B$ if and only if $x \in \mu/\lambda B$, we have

$$p_B(\lambda x) = \inf_{\mu} \{\mu > 0 : \lambda x \in \mu B\} = \lambda \inf_{\mu} \{\mu/\lambda : \mu > 0, x \in \frac{\mu}{\lambda} B\} = \lambda p_B(x),$$

which proves that p_B is positive-homogeneous. We shall now verify that the Minkowski functional is subadditive. Let x and y belong to V and $\varepsilon > 0$. There exist positive scalars λ and μ such that

$$p_B(x) < \lambda < p_B(x) + \varepsilon, \quad p_B(y) < \mu < p_B(y) + \varepsilon.$$

*If μ is a complex number and $\mu \neq 0$, then $\operatorname{sgn} \mu = |\mu|/\mu$; if $\mu = 0$, then $\operatorname{sgn} \mu = 0$.

**The property of positive homogeneity for p_B is obvious for $\lambda = 0$ since $p_B(0) = 0$.

Hence x/λ and y/μ belong to B since B is balanced. Since B is convex, we have

$$\frac{x+y}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu} \cdot \frac{x}{\lambda} + \frac{\mu}{\lambda+\mu} \cdot \frac{y}{\mu} \in B.$$

Therefore $p_B(x+y) \leq \lambda + \mu < p_B(x) + p_B(y) + 2\varepsilon$. Since ε is arbitrary, we find that p_B is subadditive:

$$p_B(x+y) \leq p_B(x) + p_B(y).$$

Now let B be balanced. We shall establish that p_B is a homogeneous function and therefore a seminorm. Since B is balanced, we have $\lambda x \in \mu B$ if and only if $x \in \mu/|\lambda|B$. Hence

$$\begin{aligned} p_B(\lambda x) &= \inf_{\mu} \{ \mu > 0 : \lambda x \in \mu B \} \\ &= |\lambda| \inf_{\mu} \left\{ \frac{\mu}{|\lambda|} : \mu > 0, x \in \frac{\mu}{|\lambda|} B \right\} = |\lambda| p_B(x), \end{aligned}$$

which was to be proved. ■

1.5. Topological Vector Spaces. Kolmogorov's Theorem

We now turn to the study of a very important concept of functional analysis, the concept of a topological vector space.

DEFINITION 20. A set V_T is called a *topological vector space* over the field P of real or complex numbers if

- 1) the set V_T is a vector space over the field P ;
- 2) the set V_T is a topological space;

3) the vector space operations are continuous in the topology of V_T , i.e., for any points x_1 and x_2 in V_T and any neighborhood Σ of the point $x_1 + x_2$ there exist neighborhoods Σ_1 and Σ_2 of the points x_1 and x_2 respectively such that $\Sigma_1 + \Sigma_2 \subset \Sigma$; and likewise if $x \in V_T$, $\alpha \in P$, and $\Sigma_{\alpha x}$ is any neighborhood of the point αx , then there exist a number $\delta > 0$ and a neighborhood Σ_x of the point x such that the relation $\beta \Sigma_x \subset \Sigma_{\alpha x}$ if $|\beta - \alpha| < \delta$ holds.

In other words a vector space V_T is a topological vector space over the field P if $V_T = (G_T, f)$, where G_T is a commutative topological group under addition and the mapping $f : P \times G_T \rightarrow G_T$ given by the rule

$$f(\alpha, x) = \alpha x$$

is continuous.

It follows directly from what has been said that the mapping $P \times P \times \cdots \times P \times G_T \times G_T \times \cdots \times G_T \rightarrow G_T$ for which

$$(\alpha_1, \alpha_2, \dots, \alpha_n, x_1, x_2, \dots, x_n) \rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

is continuous, i.e., all linear combinations of scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ and vectors x_1, x_2, \dots, x_n are continuous mappings of $P \times P \times \cdots \times P \times G_T \times G_T \times \cdots \times G_T$ into G_T .

From now on we shall abbreviate the phrase *topological vector space* to TVS.

It is clear that every normed space is a topological vector space, and a neighborhood of each point can be taken to be an open ball containing the point.

It is easy to see that a linear manifold in a TVS V_T is again a TVS if we consider the topology and the linear operations on it induced from the original space.

Since a TVS V_T is a topological space, the question naturally arises of studying the open sets in V_T .

By Lemma 1 of Sec. 1.4 the open sets in a topological space can be given by defining a neighborhood of each point. In a TVS the neighborhoods of all points can be defined by prescribing the neighborhoods of zero.

In this connection we shall establish the following property.

PROPERTY 6. *If Σ_x is a neighborhood of the point x in V_T , then $\Sigma_x - x = \Sigma_0$ is a neighborhood of zero in V_T .*

PROOF: To prove this property it obviously suffices to establish that if Σ is an open set, then $\Sigma - x_0$ is also open, for a fixed point x_0 . Indeed, let $y \in \Sigma - x_0$, i.e., $y = x - x_0$, $x \in \Sigma$. Then Σ is a neighborhood of the point $y + x_0 = x \in \Sigma$. By the continuity of addition there exist neighborhoods Σ_{x_0} and Σ_y of the points x_0 and y respectively such that $\Sigma_y + \Sigma_{x_0} \subset \Sigma$. In particular $\Sigma_y + x_0 \subset \Sigma$, i.e., $\Sigma_y \subset \Sigma - x_0$. This shows that some neighborhood Σ_y of each point $y \in \Sigma - x_0$ is contained in the set $\Sigma - x_0$, i.e., the set $\Sigma - x_0$ is open. ■

By virtue of the continuity of scalar multiplication it can be shown similarly that the set $\beta\Sigma$ is open if Σ is open and $\beta \neq 0$ belongs to the field P .

Similar properties hold for closed sets.

Thus each neighborhood of a point x in a TVS has the form $\Sigma_0 + x$, where Σ_0 is a neighborhood of the zero element of the space. If $\{\Sigma_0\}$ forms a neighborhood basis of zero (cf. Definition 3 of Sec. 1.4), then for a given point x the system $\{\Sigma_0 + x\}$ is a neighborhood basis of x .

By the definition of a neighborhood basis we find that in any TVS there exists a neighborhood basis of zero $\{\Sigma_0\}$ such that for any Σ_1 and Σ_2 in $\{\Sigma_0\}$ there exists a neighborhood $\Sigma_3 \in \{\Sigma_0\}$ possessing the property $\Sigma_3 \subset \Sigma_1 \cap \Sigma_2$.

The following properties also hold.

PROPERTY 7. *For any neighborhood $\Sigma_0 \in \{\Sigma_0\}$ there exists a neighborhood Σ such that $\Sigma + \Sigma \subset \Sigma_0$.*

PROOF: Indeed, since $0 + 0 = 0$ and addition is continuous in a TVS this property must hold. ■

PROPERTY 8. *Each neighborhood of zero Σ_0 in a TVS is an absorbing set.*

PROOF: In fact, $0 \cdot x = 0$. By the continuity of multiplication there exist a neighborhood Σ_x of the point x and a number $\delta > 0$ such that $\beta \Sigma_x \subset \Sigma_0$ for $|\beta| \leq \delta$. In particular, $x \in \frac{1}{\beta} \Sigma_0$ for $|1/\beta| \geq 1/\delta$. ■

PROPERTY 9. *Each neighborhood Σ_0 of zero in a TVS contains a balanced neighborhood of zero.*

PROOF: By the continuity of multiplication and the fact that $0 \cdot 0 = 0$ (the second factor and the right-hand side here are the zero element of the TVS), for any neighborhood of zero Σ_0 there exist a neighborhood Σ_δ and a number $\delta > 0$ such that $\beta \Sigma_\delta \subset \Sigma_0$ for $|\beta| \leq \delta$. Let $\Sigma_0^1 = \bigcup_{|\beta| \leq \delta} \beta \Sigma_\delta$.

Since $\Sigma_0^1 \supset \delta \Sigma_\delta$ and the set $\delta \Sigma_\delta$ is, as we have noted, open and therefore a neighborhood of zero, it follows that the set Σ_0^1 is also a neighborhood of zero. It is balanced: if $|\alpha| \leq 1$, then $\alpha \Sigma_0^1 = \bigcup_{|\beta| \leq \delta} \alpha \beta \Sigma_\delta \subset \Sigma_0^1$. Finally, since $\beta \Sigma_\delta \subset \Sigma_0$ for all β satisfying $|\beta| \leq \delta$, we have also $\Sigma_0^1 \subset \Sigma_0$. ■

PROPERTY 10. *Let $\{\Sigma_0\}$ be a neighborhood basis of zero in the TVS V_T . A necessary and sufficient condition for V_T to be a Hausdorff space is that $\bigcap_{\Sigma \in \{\Sigma_0\}} \Sigma = \{0\}$.*

PROOF: Let V_T be a Hausdorff space, i.e., distinct points x and y have disjoint neighborhoods. Let $x \neq 0$. Then there exists a neighborhood of the zero element $\Sigma_0 \in \{\Sigma_0\}$ not containing the point x . Consequently this intersection can contain only the zero element.

Conversely, if $\bigcap_{\Sigma \in \{\Sigma_0\}} \Sigma = \{0\}$, and $x \neq y$, then there exists a neighborhood Σ_0 not containing the point $x - y$. By Properties 7 and 9 there exists a balanced neighborhood of zero Σ such that $\Sigma + \Sigma \subset \Sigma_0$. Then $x + \Sigma$ and

$y + \Sigma$ are neighborhoods of the points x and y respectively, and they are disjoint. Indeed if they intersected and $z \in (x + \Sigma) \cap (y + \Sigma)$, we would find that

$$x - y = (z - y) - (z - x) \in \Sigma - \Sigma = \Sigma + \Sigma \subset \Sigma_0.$$

This last relation contradicts the choice of the neighborhood Σ_0 . Consequently V_T is a Hausdorff space. ■

Throughout the following we shall assume that all topological vector spaces are Hausdorff (i.e., that the Hausdorff separation axiom holds). These are the spaces of most interest in functional analysis.

We shall have need of the following proposition below.

PROPERTY 11. *If Y is a convex set in the TVS V_T , its interior $\overset{\circ}{Y}$ is convex.**

PROOF: We may assume that $\overset{\circ}{Y} \neq \emptyset$. Let $x \in Y$. We introduce the set $U = x - Y$. It is a convex set (cf. Property 3 of Section 2.1.4) and contains zero. Moreover its interior is $\overset{\circ}{U} = x - \overset{\circ}{Y}$. We shall show that the set $\overset{\circ}{U}$ is convex. To do this we shall show that $\overset{\circ}{U} = \{x : p_U(x) < 1\}$, where p_U is the Minkowski functional of the set U (cf. Property 5 of Sec. 2.1.4). We recall that the Minkowski functional of an absorbing convex set U has the property that $\{x : p_U(x) < 1\} \subset U \subset \{x : p_U(x) \leq 1\}$. Therefore for the points of $\overset{\circ}{U}$ the function $p_U(x)$ does not exceed 1. Suppose that for $x \in \overset{\circ}{U}$ we have $p_U(x) = 1$.

We shall show that any neighborhood Σ of the point x contains a point $y \notin U$, which will contradict the assumption $x \in \overset{\circ}{U}$. Indeed, since Σ is a neighborhood of the point x , there exists $\varepsilon > 0$ such that $y = (1 + \varepsilon)x \in \Sigma$. Then $p_U(y) = (1 + \varepsilon)p_U(x) = 1 + \varepsilon > 1$, which implies $y \notin U$. Consequently for points $x \in \overset{\circ}{U}$ we have $p_U(x) < 1$ and $\overset{\circ}{U} = \{x : p_U(x) < 1\}$. By Property 5 of Sec. 2.1.4 it follows that $\overset{\circ}{U}$ is convex and then $\overset{\circ}{Y} = x - \overset{\circ}{U}$ is also convex. ■

We now prove the following proposition.

PROPERTY 12. *In a TVS V_T each convex neighborhood of zero contains a balanced convex neighborhood of zero.*

PROOF: Let Σ_0 be a convex neighborhood of zero. We choose a balanced neighborhood of zero $\Sigma \subset \Sigma_0$ according to the assertion of Property 9.

*Cf. the footnote in Sec. 1.4.2. The interior of a set is the union of all open sets contained in the given set.

Since Σ is balanced, we have $\alpha^{-1}\Sigma = \Sigma$ for $\alpha \in P$, $|\alpha| = 1$, and so $\Sigma \subset \alpha_0\Sigma_0$, and therefore $\Sigma \subset U = \bigcap_{|\alpha|=1} \alpha\Sigma_0$. It follows from this that the

interior $\overset{\circ}{U}$ of the set U is a neighborhood of zero. It is clear that $\overset{\circ}{U} \subset \Sigma_0$. Being the intersection of convex sets, the set U is convex and consequently, by Property 11, $\overset{\circ}{U}$ is also convex.

We shall show that $\overset{\circ}{U}$ is balanced. To do this it suffices to establish that U is balanced. We choose λ and μ so that $0 \leq \lambda \leq 1$ and $|\mu| = 1$. Then

$$\lambda\mu U = \bigcap_{|\alpha|=1} \lambda\mu\alpha\Sigma_0 = \bigcap_{|\alpha|=1} \lambda\alpha\Sigma_0.$$

Since $\alpha\Sigma_0$ is a convex neighborhood of zero, we have $\lambda\alpha\Sigma_0 \subset \alpha\Sigma_0$. Thus $\lambda\mu U \subset U$. Consequently $\overset{\circ}{U}$ is the desired convex balanced neighborhood of zero, and $\overset{\circ}{U} \subset \Sigma_0$. ■

In topological vector spaces the concept of a bounded set is important.

DEFINITION 21. A set A in a topological vector space is called *bounded* if for any neighborhood of zero Σ_0 there exists $\beta \in P$ such that $A \subset \alpha\Sigma_0$ for all $\alpha > \beta > 0$.

We note certain simple facts involving the concept of a bounded set.

PROPOSITION 1. Let A_1 and A_2 be bounded subsets of a TVS V_T . Then the subsets $A_1 + A_2$ and λA_1 , where $\lambda \in P$, are also bounded subsets of V_T .

PROOF: For the set λA_1 the assertion is obvious. For the subset $A_1 + A_2$ the assertion follows from the fact that for every neighborhood of zero Σ_0 there exists a neighborhood of zero Σ such that $\Sigma + \Sigma \subset \Sigma_0$ (cf. Property 7 of Sec. 2.1.5). ■

PROPOSITION 2. A necessary and sufficient condition for a set A of a topological vector space to be bounded is that the relation $\alpha_n x_n \rightarrow 0$ as $n \rightarrow \infty$ hold for any sequence x_n of points of the set A and any sequence of scalars $\alpha_n \in P$ such that $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$.

PROOF: Let A be a bounded set and Σ a balanced neighborhood of zero. Then there exists $\alpha > 0$ such that $A \subset \alpha\Sigma$. Choose the number N such that $|\alpha_n|\alpha < 1$ for all $n > N$. Let $x_n \in A$. Then $\alpha^{-1}x_n \in \Sigma$, and since Σ is a balanced neighborhood, $\alpha_n\alpha^{-1}x_n = \alpha_n x_n \in \Sigma$ for all $n > N$, i.e., $\alpha_n x_n \rightarrow 0$ as $n \rightarrow \infty$.

Conversely suppose that for each sequence of points $x_n \in A$ and each sequence of scalars $\{\alpha_n\}$ such that $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, the condition $\alpha_n x_n \rightarrow 0$ as $n \rightarrow \infty$ holds. We shall prove that the set A is bounded. Suppose the contrary. Then there exists a neighborhood of zero Σ and a sequence of scalars $\gamma_n \rightarrow \infty$ such that $\gamma_n \Sigma$ does not contain A . Let $x_n \in A$ and $x_n \notin \gamma_n \Sigma$. Then none of the points $\gamma_n^{-1} x_n$ belongs to the set Σ , i.e., the sequence $\gamma_n^{-1} x_n$ does not tend to zero as $n \rightarrow \infty$. ■

In connection with the definitions given above for a topological vector space (Definition 20 of this section) and a normed space (Definition 12 of this section) the question arises as to when a TVS V_T is normed, i.e., when one can introduce a norm into V_T in such a way that the open sets defined using this norm (using open balls as in a metric space) coincide with the open sets already present in the TVS V_T .

The following theorem holds for the normability of a topological vector space.

THEOREM (Kolmogorov). *A necessary and sufficient condition for a TVS V_T to be normable is that it contain a convex bounded neighborhood of zero.*

PROOF: Let Σ be a neighborhood of the type described in the condition of the theorem. Without loss of generality we may assume that the neighborhood Σ is balanced (Property 12 above). We emphasize that according to Property 8 the neighborhood Σ is an absorbing set.

Let x be an arbitrary element of V_T . We introduce the Minkowski functional p_Σ corresponding to the set Σ ;

$$p_\Sigma(x) = \inf_{\mu} \{\mu : \mu > 0, \quad x \in \mu \Sigma\}.$$

According to the assertion of Property 5 of this section the functional p_Σ is a seminorm. We shall show that $p_\Sigma(x)$ actually defines a norm on the space V_T . To do this it suffices to show that $p_\Sigma(x) = 0$ if and only if $x = 0$. We already know that $p_\Sigma(0) = 0$. If $x \neq 0$, there exists a natural number n_0 such that $n_0 x \notin \Sigma$.

Indeed, if $y_n = nx \in \Sigma$ for any n , then according to Proposition 2 the boundedness of Σ implies the relation $\frac{1}{n} y_n \rightarrow 0$, contradicting the relation $\frac{1}{n} y_n = x \neq 0$. Consequently there exists a natural number n_0 such that $n_0 x \notin \Sigma$ and therefore $x \notin \frac{1}{n_0} \Sigma$. Thus we conclude that $p_\Sigma(x) \geq \frac{1}{n_0} > 0$, i.e., if $x \neq 0$, then $p_\Sigma(x) > 0$.

We have thus shown that the Minkowski functional corresponding to a bounded convex neighborhood Σ of zero possesses all the properties of a norm, and we can write $p_\Sigma(x) = \|x\|$.

We shall now show that the set of neighborhoods of zero of the resulting normed space coincides with the set of neighborhoods of zero already present in the TVS V_T . Let Σ_0 be an arbitrary neighborhood of zero. Since the neighborhood Σ by which the norm was introduced is a bounded set, there exists a number $r^{-1} > 0$ such that $\Sigma \subset r^{-1}\Sigma_0$. On the other hand the unit ball $\|x\| < 1$ is contained in the neighborhood Σ . Consequently the ball $\|x\| \leq r$ is contained in $r\Sigma$, i.e., in Σ_0 . Conversely suppose the ball $\|x\| \leq \rho$ is given. It follows from the definition of a norm that this ball is entirely contained in the neighborhood of zero $\rho'\Sigma$, where ρ' is an arbitrary positive number larger than ρ . This completes the proof of the sufficiency of the conditions of the theorem.

We now prove the necessity. Suppose the topology of the space V_T is normable and $\|\cdot\|$ is a norm that gives a set of open neighborhoods of zero coinciding with the set already present in V_T . Then $O = \{x : \|x\| < 1\}$ is a convex bounded neighborhood of zero. The verification requires only the fact that the unit ball is convex. Let x and y be arbitrary points of O and let $z = tx + (1-t)y$, $0 \leq t \leq 1$. Then $\|z\| = \|tx + (1-t)y\| \leq t\|x\| + (1-t)\|y\| \leq 1 + 1 - t = 1$, i.e., $z \in O$. ■

1.6. Countably Normed Spaces

Important examples of topological vector spaces are the so-called *countably normed spaces*. To define these spaces we introduce an auxiliary concept.

Suppose that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are given in the vector space V . We call these norms *consistent* if any sequence $\{x_n\}$ belonging to V , fundamental in both of them, and convergent to the limit $x \in V$ in one of the norms also converges to the same limit in the second norm.

DEFINITION 22. A vector space V is called a *countably normed space* if a countable system of mutually consistent norms $\|\cdot\|_n$, $n = 1, 2, \dots$ is defined on it.

Every countably normed space becomes a topological vector space if a neighborhood basis of zero is taken to be the set of neighborhoods $\Sigma_{n,\varepsilon}$ depending on the index n and the number $\varepsilon > 0$, each neighborhood $\Sigma_{n,\varepsilon}$ consisting of the elements $x \in V$ for which

$$\|x\|_1 < \varepsilon, \dots, \|x\|_n < \varepsilon.$$

It is easy to see that such a system of neighborhoods of zero indeed defines a topology (cf. Lemma 1 of Sec. 1.4.1). Indeed zero belongs to each

neighborhood, and the intersection of two neighborhoods $\Sigma_{n_1, \varepsilon_1}$ and $\Sigma_{n_2, \varepsilon_2}$ of this type is again a neighborhood of the same type (one must take the smaller of the numbers ε_1 and ε_2 and the larger of the numbers n_1 and n_2). Finally for any neighborhood $\Sigma_{n, \varepsilon}$ there exists another neighborhood contained in it.

It is verified just as easily that the operations of vector addition and scalar multiplication by elements of the field P are continuous in the given topology. Let us verify, for example, that the operation of scalar multiplication is continuous. For any neighborhood $\Sigma_{n, \varepsilon}$ in view of the relation $0 \cdot 0 = 0$ (the second factor on the left and the right-hand side are the zero element of the space), there exist a neighborhood of zero $\Sigma_{n_1, \varepsilon_1}$ and a number $\delta > 0$ such that

$$\alpha \Sigma_{n_1, \varepsilon_1} \subset \Sigma_{n, \varepsilon} \quad \text{for } |\alpha| \leq \delta;$$

to obtain this relation it suffices to set $n_1 = n$, $\varepsilon_1 = \varepsilon$, and $\delta \leq 1$.

We emphasize that every countably normed space satisfies the first axiom of countability, since the neighborhood system at zero $\Sigma_{n, \varepsilon}$, $n = 1, 2, \dots$, $\varepsilon > 0$, can be replaced by the subsystem $\Sigma_{n, 1/k}$, $n = 1, 2, \dots$, $k = 1, 2, \dots$. The topology is not altered when this is done. Thus in this space convergent sequences "recover their rights," and the topology can be described in terms of them.

In a countably normed space V one can introduce a metric, for example, by the rule

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{\|x - y\|_n}{2^n (1 + \|x - y\|_n)}, \quad x, y \in V.$$

The metric $\rho(x, y)$ possesses the interesting property of translation invariance:

$$\rho(x + z, y + z) = \rho(x, y), \quad x, y, z \in V.$$

EXAMPLES

1. Consider the space $K[a, b]$ of infinitely differentiable functions f on the closed interval $[a, b]$ with the usual linear operations on functions. Define

$$\|f\|_n = \sup_{\substack{a \leq t \leq b \\ 0 \leq k \leq n}} |f^{(k)}(t)|.$$

Obviously all these norms are mutually consistent, and the topology of a countably normed space is thereby introduced.

2. Let S be the space of infinitely differentiable functions on the line that tend to zero at infinity along with all their derivatives faster than any power of $|t|^{-1}$, i.e. $t^k f^{(q)}(t) \rightarrow 0$ as $|t| \rightarrow \infty$ for any fixed k and q .

We define

$$\|f\|_n = \sup_{k,q \leq n-1} |t^k f^{(q)}(t)|, \quad n = 1, 2, \dots$$

It is obvious that S is a countably normed space.

In conclusion we note that the norms in a countably normed space V can be assumed to satisfy the condition

$$\|x\|_k \leq \|x\|_m \quad \text{for } k < m,$$

since otherwise we would consider the norms

$$\|x\|_{k,1} = \max(\|x\|_1, \|x\|_2, \dots, \|x\|_k),$$

which defines the same topology.

A countably normed space can be completed on the metric ρ introduced above. We note that when this is done, a sequence $\{x_n\}$ is fundamental with respect to the metric ρ (resp. converges in the metric ρ) if and only if it is fundamental with respect to each of the metrics $\|\cdot\|_n$ (resp. converges in each of the norms), i.e., the completeness of V means that each sequence in V that is fundamental in each of the norms $\|\cdot\|$ converges.

Completing the space V on each of the norms $\|\cdot\|_k$, which satisfy the inequality $\|x\|_k \leq \|x\|_m$ for $k < m$, we obtain a natural inclusion of complete normed spaces

$$V_k \supset V_m \quad \text{for } k < m,$$

where V_k and V_m are normed spaces with the norms $\|\cdot\|_k$ and $\|\cdot\|_m$ respectively, and $\bigcap_{k=1}^{\infty} V_k \supset V$.

EXERCISES

1. Let $V_T = \mathbf{C}$ be a one-dimensional vector space over the field of complex numbers \mathbf{C} . Prove that the balanced sets are the following: \mathbf{C} , \emptyset , the one-point set $\{0\}$, and a disk (open or closed) with center at the point 0. Find the balanced sets if $V_T = \mathbf{R}^2$ (a two-dimensional space of the field of real numbers \mathbf{R}^1).

Let $B = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq |z_2|\}$, $\mathbf{C}^2 = \mathbf{C} \times \mathbf{C}$, where \mathbf{C} is the field of complex numbers. Prove that B is a balanced set.

2. Let M be a subspace (a closed linear manifold) of the normed vector space N not coinciding with N . Then for any $\varepsilon > 0$ there exists a $y \in N$ with norm equal to 1 such that $\|x - y\| > 1 - \varepsilon$ for all $x \in M$.

3. Prove that the closure of a linear manifold in a TVS is a linear manifold.

4. Prove that the closure of a convex set is convex and that the closure of an absolutely convex set is absolutely convex.

5. Let V be a vector space containing a distinguished family $\{\Sigma_0\}$ satisfying the following conditions: for any neighborhood $\Sigma_0 \in \{\Sigma_0\}$ there exists $\Sigma \in \{\Sigma_0\}$ such that $\Sigma + \Sigma \subset \Sigma_0$; each $\Sigma_0 \in \{\Sigma_0\}$ is a balanced and absorbing set; for any $\Sigma_1, \Sigma_2 \in \{\Sigma_0\}$ there exists $\Sigma_3 \in \{\Sigma_0\}$ such that $\Sigma_3 \subset \Sigma_1 \cap \Sigma_2$. Then if we take the neighborhoods of a point x to be any sets of the form $x + \Sigma_0$, $\Sigma_0 \in \{\Sigma_0\}$, the vector space V becomes a topological vector space in which the system $\{\Sigma_0\}$ is a neighborhood basis of zero (cf. Properties 7, 8, and 9 of Sec. 2.1.5).

6. Let $X = \{x(t)\}$ be the set of functions defined on the real line \mathbf{R}^1 , infinitely differentiable on it and vanishing outside some closed interval (different for each individual function). The sum of two functions and the product of a function by a number are defined in the usual way. Neighborhoods of zero are taken to be the sets $\Sigma_{(k,\varepsilon)} = \{x(t) : |x^{(i)}(t)| < \varepsilon, \quad i = 1, \dots, k\}$ for any $\varepsilon > 0$ and any positive integer k . Verify that all the axioms of a topological vector space are satisfied.

7. A subset B of a topological vector space V_T is called *totally bounded* if for any neighborhood of zero Σ_0 in V_T there exists a finite set $\{x_k\}_{k=1}^n \subset B$ such that $B \subset \bigcup_{k=1}^n (x_k + \Sigma_0)$. Prove that every finite set is totally bounded and that a totally bounded set is bounded.

8. A topological vector space is called *locally convex* if it has a basis of convex neighborhoods of zero. Prove that the set of continuous functions on the closed interval $[a, b]$ becomes a locally convex space when endowed with the norm $\|x\| = \sup_{a \leq t \leq b} |x(t)|$, i.e., the space $C[a, b]$ (and similarly $C(X)$ for any compact set X) is a locally convex topological vector space.

9. Prove that the function ρ introduced on the countably normed space V (cf. Sec. 2.1.5) satisfies all the axioms of a distance.

2. BOUNDED LINEAR TRANSFORMATIONS ON BANACH AND F -SPACES. BASIC PRINCIPLES OF FUNCTIONAL ANALYSIS

In this section we study for the most part the properties of transformations on Banach spaces. Many of the propositions, facts, and definitions

given below hold in more general spaces: topological vector spaces, F -spaces and others. The corresponding generalizations are carried out in the course of the discussion. We emphasize that although we are not striving to discuss the material in maximal generality from the outset, nevertheless if the proof of a theorem is conceptually simpler in the more general situation, we give it in that form immediately.

2.1. Bounded Linear Transformations on Banach Spaces.

The Banach Space of Transformations.

The Concept of an F -Space.

Let V_{T_1} and V_{T_2} be two topological vector spaces and let the linear transformation A map V_{T_1} into V_{T_2} , i.e., $A : V_{T_1} \rightarrow V_{T_2}$.

DEFINITION 1. A linear transformation $A : V_{T_1} \rightarrow V_{T_2}$ is called *bounded* if it takes bounded sets into bounded sets.* (We remark that this definition also applies to linear functionals.

It is obvious that if a linear transformation A maps one normed space N_1 into another N_2 , i.e., $A : N_1 \rightarrow N_2$, then the definition just given is equivalent to the following.

DEFINITION 1'. A linear transformation $A : N_1 \rightarrow N_2$ is called *bounded* if there exists a constant M such that $\|Ax\|_{N_2} \leq M\|x\|_{N_1}$ for any $x \in N_1$.

Here N_1 and N_2 are two normed spaces and the notation $\|Ax\|_{N_2}$ and $\|x\|_{N_1}$ means that the norms are taken in the spaces N_2 and N_1 respectively. Whenever no confusion can arise we shall omit the subscripts N_1 and N_2 .

The following theorem holds.

THEOREM 1. Let N_1 and N_2 be two normed spaces and A a linear transformation from N_1 to N_2 . Then a necessary and sufficient condition for A to be continuous is that it be bounded.

PROOF. NECESSITY: Suppose a continuous transformation (cf. Definition 12 of Sec. 1.2) is not bounded. Then there exists a sequence of elements $\{x_n\}$, $x_n \in N_1$, such that $\|Ax_n\| > n\|x_n\|$. Let $\xi_n = \frac{x_n}{n\|x_n\|}$. Then $\|\xi_n\| = 1/n \rightarrow 0$, as $n \rightarrow \infty$, i.e., $\xi_n \rightarrow 0$ as $n \rightarrow \infty$. (We recall that all the definitions we have given for metric spaces carry over to normed spaces. In

*A set U situated in a normed space is *bounded* if there exists a number $N > 0$ such that $\|x\| = \rho(x, 0) \leq N$ for any point $x \in U$ (cf. Exercise 5 in Sec. 1.2).

particular $\xi_n \rightarrow 0$ as $n \rightarrow \infty$ means that $\rho(0, \xi_n) \rightarrow 0$, i.e., $\|\xi_n\| \rightarrow 0$, as $n \rightarrow \infty$.)

Let us calculate the quantity $\|A\xi_n\|$. We have $\|A\xi_n\| = \frac{1}{n\|x_n\|}\|Ax_n\| >$

1. Therefore $\|A\xi_n\|$ does not tend to zero as $n \rightarrow \infty$, which means that $A\xi_n$ does not tend to zero as $n \rightarrow \infty$. Now the transformation A is linear (cf. Definition 9 of Sec. 2.1.2), so that $A0 = A(x - x) = Ax - Ax = 0$. It therefore follows from the preceding that the sequence $\{A\xi_n\}$ does not tend to the element $A0$, contradicting the continuity of the transformation A . Consequently our assumption that the transformation A is not bounded is false.

SUFFICIENCY: Suppose the transformation A is bounded. We shall prove it is continuous. Let $x_n \rightarrow x$, i.e., $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. Then $\|Ax_n - Ax\| = \|A(x_n - x)\| \leq M\|x_n - x\| \rightarrow 0$. Thus $Ax_n \rightarrow Ax$, and so the transformation A is continuous. ■

(We remark also that the theorem naturally remains true for linear functionals satisfying the hypotheses of the theorem.)

For a linear transformation A of one topological vector space V_{T_1} into another V_{T_2} we have the following theorem.

THEOREM 2. *Let V_{T_1} and V_{T_2} be two topological vector spaces and A a linear transformation from V_{T_1} into V_{T_2} . If A is continuous, then it is bounded.*

PROOF: Let E be a bounded subset of V_{T_1} (cf. Definition 21, Sec. 2.1.5) and Σ_0 a neighborhood of zero in V_{T_2} . Since A is a continuous linear transformation, we find first of all that the element 0 maps to 0, and second that in V_{T_1} there is a neighborhood of zero Σ such that $A(\Sigma) \subset \Sigma_0$. Since the set E is bounded, we have $E \subset \alpha\Sigma$ for sufficiently large α . Consequently

$$A(E) \subset A(\alpha\Sigma) = \alpha A(\Sigma) \subset \alpha\Sigma_0,$$

i.e., $A(E)$ is a bounded subset of V_{T_2} . ■

It is not difficult to show that in general boundedness of A in a TVS does not imply that A is continuous.

DEFINITION 2. Let A be a bounded linear transformation from normed space N_1 into a normed space N_2 . The *norm* of the transformation A , denoted $\|A\|$, is the smallest constant M satisfying the condition $\|Ax\| \leq M\|x\|$.

Thus by definition the norm $\|A\|$ of the transformation A possesses the following two properties:

a) $\|Ax\| \leq \|A\| \|x\|$ for any $x \in N_1$;

b) for any $\varepsilon > 0$ there exists an element x_ε such that the inequality $\|Ax_\varepsilon\| > (\|A\| - \varepsilon)\|x_\varepsilon\|$ holds.

We shall show that $\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$, or, what is the same, that

$$\|A\| = \sup_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}.$$

Indeed if $\|x\| \leq 1$, then $\|Ax\| \leq \|A\| \|x\| \leq \|A\|$. Thus $\sup_{\|x\| \leq 1} \|Ax\| \leq \|A\|$.

On the other hand, for any $\varepsilon > 0$ there exists an element x_ε such that $\|Ax_\varepsilon\| > (\|A\| - \varepsilon)\|x_\varepsilon\|$. Let $\xi_\varepsilon = \frac{x_\varepsilon}{\|x_\varepsilon\|}$. Then $\|A\xi_\varepsilon\| = \frac{\|Ax_\varepsilon\|}{\|x_\varepsilon\|} > \frac{1}{\|x_\varepsilon\|} (\|A\| - \varepsilon)\|x_\varepsilon\| = \|A\| - \varepsilon$. Since $\|\xi_\varepsilon\| = 1$, we have $\sup_{\|x\| \leq 1} \|Ax\| \geq \|A\xi_\varepsilon\| \geq \|A\| - \varepsilon$. Consequently $\sup_{\|x\| \leq 1} \|Ax\| \geq \|A\|$, and $\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$.

NOTE. It follows from this reasoning that $\|A\| = \sup_{\|x\|=1} \|Ax\|$, and so

$$\|A\| = \sup_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}.$$

The space $(V_1 \rightarrow V_2)$ of transformations mapping the vector space V_1 into the vector space V_2 was introduced above (cf. Definition 10 of Sec. 2.1.2). This space plays an important role in various parts of analysis, and we now continue our study of it.

We now assume that the vector spaces V_1 and V_2 given previously are normed. For convenience, and to emphasize that they are normed, we denote them by N_1 and N_2 respectively. The vector space whose elements are the bounded linear transformations of N_1 into N_2 is now denoted $(N_1 \rightarrow N_2)$. A norm can be introduced into the space $(N_1 \rightarrow N_2)$. To do this we define the norm of an element A of $(N_1 \rightarrow N_2)$ by the rule $\|A\| = \sup_{\|x\|=1} \|Ax\|$. It is

easy to see that this norm satisfies the axioms of Definition 12 of Sec. 2.1.3. Thus the vector space $(N_1 \rightarrow N_2)$ whose elements are the bounded linear transformations from N_1 into N_2 is a normed vector space. The question naturally arises: When is this space complete, i.e., a Banach space?

The answer to this question is contained in the following theorem.

THEOREM 3. *If a normed vector space B_2 is a Banach space, then the space of bounded linear transformations $(N_1 \rightarrow B_2)$ is also a Banach space.*

PROOF: Let $\{A_n\}$ be a fundamental sequence in the space of transformations $(N_1 \rightarrow B_2)$, i.e., $\|A_n - A_m\| \rightarrow 0$ as $n, m \rightarrow \infty$. For any x in N_1 we have $\|A_n x - A_m x\| \leq \|A_n - A_m\| \|x\| \rightarrow 0$ as $n, m \rightarrow \infty$. Therefore for each fixed $x \in N_1$ the sequence $\{A_n x\}$ is fundamental in B_2 , and so, since the space B_2 is complete, this sequence converges. Let $y = \lim_{n \rightarrow \infty} A_n x$. We have thus obtained a mapping from N_1 into B_2 . We denote the transformation that this mapping defines by A . It follows from properties of the limit that A is linear. We shall show that it is bounded. It follows from the relation $\|A_n - A_m\| \rightarrow 0$ as $n, m \rightarrow \infty$ that $\|\|A_n\| - \|A_m\|\| \rightarrow 0$ as $n, m \rightarrow \infty$, i.e., that the numerical sequence $\{\|A_n\|\}$ is fundamental in \mathbf{R}^1 , hence bounded. There exists a constant M such that $\|A_n\| \leq M$ for any natural number n . We obtain from this that $\|A_n x\| \leq \|A_n\| \|x\| \leq M \|x\|$, i.e., by virtue of the fact that the function defining the norm (distance) is continuous, we have

$$\|Ax\| = \lim_{n \rightarrow \infty} \|A_n x\| \leq M \|x\|.$$

Thus the transformation A is bounded. It has been defined as a mapping from N_1 into B_2 by the rule given above. We shall now show that A is the limit of the sequence $\{A_n\}$ in the sense of convergence in the norm of the space $(N_1 \rightarrow B_2)$. Let $\varepsilon > 0$ and choose n_0 so that $\|A_{n+p} x - A_n x\| < \varepsilon$ for $n \geq n_0$, $p > 0$, and any x such that $\|x\| \leq 1$. Let $p \rightarrow \infty$. Then $\|Ax - A_n x\| \leq \varepsilon$ for $n \geq n_0$ and all x of norm not greater than 1. Therefore for $n \geq n_0$

$$\|A_n - A\| = \sup_{\|x\| \leq 1} \|(A_n - A)x\| \leq \varepsilon.$$

Consequently $A = \lim_{n \rightarrow \infty} A_n$ in the sense of convergence in the norm of the space $(N_1 \rightarrow B_2)$, i.e., the latter is a Banach space. ■

DEFINITION 3. Let the continuous linear functionals F map the topological vector space V_T into the coefficient field P , i.e. $F : V_T \rightarrow P$. Then the dual space to V_T is defined to be the space $\{F\}$ of all such functionals.

It follows from the theorem proved above that, because the field P is complete the dual space to a normed space is a Banach space.

EXAMPLES

1. The norm of the identity operator $E : N \rightarrow N$ (i.e., the operator that assigns to each element x of the normed space N the same element x of N) is obviously 1. Indeed $Ex = x$ and so $\sup_{\|x\|=1} \|Ex\| = \sup_{\|x\|=1} \|x\| = 1$. Similarly the norm of the zero transformation $0 : N_1 \rightarrow N_2$ (i.e., the transformation defined by the rule $0x = 0$ for any $x \in N_1$) is zero.

2. Consider the space of continuous functions $C[a, b]$ and the differentiation operator $D : Df(t) = f'(t)$, for $f \in C[a, b]$, $a \leq t \leq b$. This operator, which we take to be a mapping of $C[a, b]$ into $C[a, b]$, is not defined on the entire space of continuous functions but only on the linear manifold of functions having a continuous derivative. The operator D is obviously linear, but not bounded. Indeed, if it were bounded, then according to Theorem 1 of this section it would be continuous. However the sequence $f_n = \frac{\sin nt}{n}$ converges to zero in the metric of $C[a, b]$ (since $\max_{a \leq t \leq b} \frac{|\sin nt|}{n} \rightarrow 0$ as $n \rightarrow \infty$), while the sequence $Df_n = \cos nt$ does not tend to zero. The transformation D , regarded as a transformation from $C^1[a, b]$ into $C[a, b]$, is bounded.

3. Consider the functional F in the space $C[0, 1]$ acting according to the rule $F(f(t)) = f(0)$, $0 \leq t \leq 1$. The norm of this functional is obviously 1. Indeed $\sup_{\|f\|=1} |F(f)| = \|F\| = \sup_{\max_{0 \leq t \leq 1} |f(t)|=1} |f(0)| = 1$. Similarly the norm

of the functional $F(f) = \int_0^1 f(t)g(t) dt$, $g \in C[0, 1]$ (this functional is also defined in $C[0, 1]$) is calculated from the formula $\|F\| = \int_0^1 |g(t)| dt$.

2.2. The Uniform Boundedness Principle

It is also natural to ask whether the space $(N_1 \rightarrow N_2)$ is complete in the sense of pointwise convergence. We shall give an answer to that question below.

We first prove the following auxiliary proposition.

PROPOSITION 1. *Suppose a sequence of bounded linear transformations $\{A_n\} \in (N_1 \rightarrow N_2)$, $n = 1, 2, \dots$, is given such that the numerical sequence $\{\|A_n\|\}$, $n = 1, 2, \dots$, is not bounded. Then the set $\{\|A_n x\|\}$ is not bounded for x in any closed ball $K(x_0, \varepsilon)$.*

PROOF: Suppose the contrary, i.e., that the set $\{\|A_n x\|\}$ is bounded in some ball $K(x_0, \varepsilon)$. Obviously the element $x = \frac{\varepsilon}{\|\xi\|} \xi + x_0 \in K(x_0, \varepsilon)$, where ξ is an arbitrary element of N_1 . For a normed linear space N_2 the boundedness of the set $\{\|A_n x\|\}$, $x \in K(x_0, \varepsilon)$, $n = 1, 2, \dots$ means that there exists a constant C such that $\|A_n x\| \leq C$ for any $x \in K(x_0, \varepsilon)$ and any n . Consequently

$$\left| \frac{\varepsilon}{\|\xi\|} \|A_n \xi\| - \|A_n x_0\| \right| \leq \left\| \frac{\varepsilon}{\|\xi\|} A_n \xi + A_n x_0 \right\| \leq C.$$

From these inequalities we obtain

$$\|A_n \xi\| \leq \frac{C + \|A_n x_0\|}{\varepsilon} \|\xi\|.$$

By hypothesis the sequence $\{\|A_n x_0\|\}$ is bounded by the constant C . Consequently there exists a constant $C_1 = 2C/\varepsilon$ such that $\|A_n \xi\| \leq C_1 \|\xi\|$ for any n , i.e., $\|A_n\| \leq C_1$, contradicting the hypothesis. ■

The following theorem has great significance in the theory of linear transformations. It is regarded as one of the fundamental principals of functional analysis, the so-called *uniform boundedness principle*.

THEOREM 4 (Banach). *Suppose a sequence $\{A_n\}$ of bounded linear transformations mapping a Banach space B into a normed space N converges pointwise as $n \rightarrow \infty$ to an operator A . Then the numerical sequence $\{\|A_n\|\}$ is bounded. Consequently $\lim_{n \rightarrow \infty} A_n x = 0$ uniformly on $n = 1, 2, 3, \dots$ and the transformation $Ax = \lim_{n \rightarrow \infty} A_n x$ is bounded.*

PROOF: Suppose, to the contrary, that the numerical sequence $\{\|A_n\|\}$ is not bounded. According to Proposition 1 the set $\{\|A_n x\|\}$ is not bounded in any closed ball $K(x_0, \varepsilon)$. Let $K_0 = K_0(x_0, \varepsilon_0)$ be some closed ball in K . Then the set $\{\|A_n x\|\}$, $x \in K_0$, is not bounded. Therefore there exists an index n_1 and an element $x_1 \in K_0$ such that $\|A_{n_1} x_1\| > 1$. The operator A_{n_1} is a bounded linear operator. Therefore according to Theorem 1 of this section it is also continuous. Consequently there exists a ball $K_1 = K_1(x_1, \varepsilon_1) \subset K_0$ such that for all $x \in K_1$ the inequality $\|A_{n_1} x\| > 1$ holds. In fact, according to Example 7 at the end of Sec. 1.2, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|A_{n_1} x_1 - A_{n_1} x\| < \varepsilon$ if $\|x - x_1\| < \delta$. Therefore

$$\| \|A_{n_1} x_1\| - \|A_{n_1} x\| \| \leq \|A_{n_1} x_1 - A_{n_1} x\| < \varepsilon,$$

i.e.,

$$-\varepsilon < \|A_{n_1} x_1\| - \|A_{n_1} x\| < \varepsilon,$$

whence

$$\|A_{n_1} x\| > \|A_{n_1} x_1\| - \varepsilon.$$

Choosing ε so small that the inequality $\|A_{n_1} x_1\| - \varepsilon > 1$ holds, we find $\delta > 0$ such that for all x in the ball $\|x - x_1\| < \delta$ the inequality $\|A_{n_1} x\| > 1$ holds. Finally we choose the radius ε_1 of the closed ball K_1 smaller than δ and we choose $K_1 \subset K_0$.

Continuing this process we construct a nested sequence of closed balls $K_0 \supset K_1 \supset K_2 \supset \dots \supset K_n \supset \dots$ whose radii $\varepsilon_0, \varepsilon_1, \dots$ can always be

assumed to satisfy the inequalities $\varepsilon_0 > \varepsilon_1 > \dots \rightarrow 0$. Then according to the nested ball principle (Theorem 2 of Sec. 1.3) there exists a point \bar{x} belonging to all the balls K_n . At this point the relation $\|A_{n_k} \bar{x}\| \geq k$ holds, contradicting the fact that the sequence $\{A_n x\}$ converges for any x in the Banach space B . Thus our assumption that the numerical sequence $\{\|A_n\|\}$ is not bounded is false. Consequently there exists a constant M such that $\|A_n x\| \leq M\|x\|$ for all indices n and any x . Passing to the limit as $n \rightarrow \infty$, we find that $\|Ax\| \leq M\|x\|$, i.e., the transformation A is bounded. ■

We remark that the proof of the theorem does not change, and the theorem remains valid, if instead of pointwise convergence of the sequence $\{A_n\}$ we require that the sequence $A_n x$ should be fundamental at each point $x \in B$ or even if we require only that it be bounded at each point. Then the theorem just proved can be also stated as follows (*the principle of condensation of singularities*): if $\sup_n \|A_n\| = \infty$, then there exists an element $x_0 \in B$ such that $\sup_n \|A_n x_0\| = \infty$.

As an application of the theorem just proved we shall show that there exist continuous functions $f(x)$ on the closed interval $[-\pi, \pi]$ for which the Fourier series $\sum_k c_k e^{ikx}$, $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt$ diverges at some points.

It is known from the theory of trigonometric series that the partial sum of a Fourier series can be written in the form $S_n(x) = \int_{-\pi}^{\pi} f(x+z) D_n(z) dz$,

where $D_n(z) = \frac{1}{2\pi} \frac{\sin \frac{2n+1}{2} z}{\sin \frac{z}{2}}$ is called the *Dirichlet kernel*. We remark first

of all that $\int_{-\pi}^{\pi} |D_n(z)| dz \rightarrow \infty$ as $n \rightarrow \infty$. Indeed, the numerator of the

fraction $|D_n(z)| = \frac{|\sin \frac{2n+1}{2} z|}{2\pi |\sin \frac{z}{2}|}$ assumes the value 1 at the points where

$\frac{2n+1}{2} z = (k + \frac{1}{2})\pi$, $k = 0, 1, \dots, n$. We surround each of these points with

the intervals on which $|\frac{2n+1}{2} z - \frac{2k+1}{2} \pi| < \frac{\pi}{3}$. Each of these intervals

has length $\frac{4\pi}{3(2n+1)}$. In each of these intervals $|\sin \frac{2n+1}{2} z|$ is at least 1/2.

Let us estimate the quantity $\sin \frac{z}{2}$ on the k th interval ($k = 0, 1, \dots, n$). We

have $\sin \frac{z}{2} < \frac{z}{2} < \frac{1}{2} \left(\frac{2k+1}{2} \pi + \frac{\pi}{3} \right) \left(\frac{2n+1}{2} \right)^{-1} < \frac{k+1}{2n+1} \pi$. Therefore the integral of $|D_n(z)|$ taken only over these intervals is larger than the sum

$$\frac{1}{2\pi} \sum_{k=0}^n \frac{1}{2} \frac{1}{\frac{k+1}{2n+1} \pi} \frac{4\pi}{3(2n+1)} = \frac{1}{3\pi} \sum_{k=0}^n \frac{1}{k+1}.$$

This sum tends to infinity as $n \rightarrow \infty$. Consider the sequence of functionals

$F_n(f) = \int_{-\pi}^{\pi} D_n(x)f(x) dx$ in the complete normed space $C[-\pi, \pi]$. According to Exercise 3 of the preceding section we have $\|F_n\| = \int_{-\pi}^{\pi} |D_n(x)| dx$. According to what has been said, the norms of the functionals F_n are not bounded, and consequently by the theorem just proved there exists a continuous function $f(x)$ on $[-\pi, \pi]$ such that $\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} D_n(x)f(x) dx$ does not exist. Therefore the Fourier series of that function is divergent at zero.

We shall now prove a theorem that answers the question of the completeness of the space of linear transformations in the sense of pointwise convergence.

THEOREM 5. *Let B_1 and B_2 be Banach spaces. Then the space of bounded linear transformations ($B_1 \rightarrow B_2$) is complete in the sense of pointwise convergence.*

PROOF: Consider some point $x \in B_1$ and a sequence $\{A_n\}$ that is fundamental in the sense of pointwise convergence. By the completeness of B_2 there exists an element $y \in B_2$ such that $y = \lim_{n \rightarrow \infty} A_n x$. This defines a linear transformation $y = Ax$ mapping the space B_1 into the space B_2 . It follows obviously from properties of the limit that A is a linear transformation. We then conclude from Theorem 4 of this section that the transformation A is also bounded. ■

Theorem 4 (the uniform boundedness principle) admits a simple generalization to the so-called F -spaces, where it goes by the name of the *principle of equicontinuity*. In this connection we give the following definition.

DEFINITION 4. A vector space with metric ρ is called an F -space if the following conditions are met:

1. The metric ρ is translation-invariant, i.e.,

$$\rho(x, y) = \rho(x - y, 0).$$

2. The mapping $f : P \times F \rightarrow F$ defined by the rule $f(\alpha, x) = \alpha x$ is continuous in $\alpha \in P$ for each $x \in F$ and continuous in x for each α , where P is the field over which the vector space is defined.

3. The metric space F is complete.

It will be shown later that an F -space is a topological vector space, and therefore it is natural to call a set A in an F -space *bounded* if for each neighborhood of zero Σ_0 there exists $\beta \in P$ such that $A \subset \alpha \Sigma_0$ for all $\alpha > \beta > 0$.

THEOREM 6 (Principle of equicontinuity). *Suppose for each element σ of some set Σ a continuous linear transformation A_σ is defined mapping one*

F-space F_1 into another F -space F_2 . If the set $\{A_\sigma x : \sigma \in \Sigma\}$ is bounded for each $x \in F_1$, then $\lim_{x \rightarrow 0} A_\sigma x = 0$ uniformly with respect to $\sigma \in \Sigma$.

In connection with this theorem we make a several clarifications. There are two hypotheses in the statement of the theorem, in each of which one of the two parameters is fixed. The theorem asserts a relation that holds when both parameters vary. Since an F -space is a particular kind of metric space, the concept of limit and the concept of a variable tending to a given point, etc. are understood in the sense of a metric space.

PROOF: Fix a number $\varepsilon > 0$, and for any natural number n consider the set

$$X_n = \left\{ x : \rho\left(\frac{1}{n}A_\sigma x, 0\right) \leq \varepsilon, \quad \sigma \in \Sigma \right\}, \quad n = 1, 2, \dots$$

The mapping A_σ is continuous, and so the sets X_n are closed for any n .

In fact the distance function $\rho(x, y)$ is a continuous function of the argument x for fixed y (cf. Example 2 at the end of Sec. 1.2.4).

Therefore if $x_i \in X_n$ and $x_i \rightarrow x$ as $i \rightarrow \infty$, then $\rho\left(\frac{1}{n}A_\sigma x_i, 0\right) \leq \varepsilon$, and consequently $\lim_{i \rightarrow \infty} \rho\left(\frac{1}{n}A_\sigma x_i, 0\right) = \rho\left(\frac{1}{n}A_\sigma x, 0\right) \leq \varepsilon$, i.e., $x \in X_n$.

Thus the sets X_n are closed, since they contain all their limit points.

We now continue with the proof of the theorem. We shall show that the F -space F_1 admits the representation $F_1 = \bigcup_{n=1}^{\infty} X_n$. In fact for each $x \in F_1$ the set $\{A_\sigma x\}$ is bounded in the F -space F_2 , and so $\rho\left(\frac{1}{n}A_\sigma x, 0\right) \rightarrow 0$ as $n \rightarrow \infty$ for any $x \in F_1$. In other words if $x \in F_1$, there exists an index n such that $x \in X_n$.

By the Baire category theorem (Theorem 3 of Sec. 1.3) at least one of the sets X_n , say X_{n_0} , is dense in some open set. Therefore there exists a ball $O(x_0, \delta) \subset X_{n_0}$ consisting entirely of points of the set X_{n_0} , i.e., the inequality $\rho\left(\frac{1}{n_0}A_\sigma x, 0\right) \leq \varepsilon$ if $\rho(x, x_0) = \rho(x - x_0, 0) < \delta$ (cf. the remark in Sec. 1.2.3). Let $x - x_0 = y$. Then for any $\sigma \in \Sigma$ we have $\rho\left(\frac{1}{n_0}A_\sigma(y + x_0), 0\right) \leq \varepsilon$, if $\rho(y, 0) < \delta$. We write the relations

$$\begin{aligned} \rho\left(\frac{1}{n_0}A_\sigma y, 0\right) &= \rho\left(\frac{1}{n_0}A_\sigma y + \frac{1}{n_0}A_\sigma x_0 - \frac{1}{n_0}A_\sigma x_0, 0\right) \\ &= \rho\left(\frac{1}{n_0}A_\sigma(y + x_0) - \frac{1}{n_0}A_\sigma x_0, 0\right) = \rho\left(\frac{1}{n_0}A_\sigma(y + x_0), \frac{1}{n_0}A_\sigma x_0\right) \\ &\leq \rho\left(\frac{1}{n_0}A_\sigma(y + x_0), 0\right) + \rho\left(\frac{1}{n_0}A_\sigma x_0, 0\right) \leq 2\varepsilon. \end{aligned}$$

Therefore, if $\rho(y, 0) < \delta$, then $\rho\left(\frac{1}{n_0}A_\sigma y, 0\right) = \rho\left(A_\sigma \frac{1}{n_0}y, 0\right) \leq 2\varepsilon$ for all $\sigma \in \Sigma$. Any point $x \in F_1$ can be represented in the form $x = n_0 \cdot \frac{x}{n_0} = \frac{z}{n_0}$,

$z = n_0 x$. The last inequality holds for any y and, in particular, for $y = n_0 x$. Therefore $\rho(A_\sigma x, 0) \leq \varepsilon$ if $\rho(n_0 x, 0) \leq \delta$. If $x \rightarrow 0$, then $\rho(n_0 x, 0) \rightarrow 0$, and consequently $\rho(A_\sigma x, 0) \rightarrow 0$ uniformly on $\sigma \in \Sigma$. ■

As a corollary of this theorem we have the following.

PROPOSITION 2. *Every F -space is a topological vector space.*

PROOF: An F -space is a vector space. The continuity of vector addition follows from the definition of an F -space. We need to verify that the mapping of $P \times F$ into F given by $f(\alpha, x) = \alpha x$ is continuous. This mapping is linear in the two variables, continuous in α for each fixed x , and continuous in x for each fixed α . For each fixed x_0 consider the set $\{\alpha x_0\}$, $|\alpha| < 1$. This set is bounded. Indeed, if β is an element of the scalar field (the real or complex numbers) and sufficiently small in absolute value, the scalar $\alpha\beta$ will also be arbitrarily small. Using the continuity of the mapping $f(\alpha, x)$ in α for fixed $x = x_0$, we conclude that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(\alpha x_0, 0) < \varepsilon$ if $|\alpha| < \delta$. Choosing $\beta \in P$ sufficiently small that $|\alpha\beta| < \delta$, we find that $\rho(\beta\alpha x_0, 0) < \varepsilon$, i.e., $\beta\alpha x_0$ belongs to an arbitrarily small ball with center at zero. This means that $\beta\{\alpha x_0\} \subset \Sigma_0$, where Σ_0 is an arbitrary neighborhood of zero, i.e., the set $\{\alpha x_0\}$ is bounded. According to Theorem 6 applied to the mapping $f(\alpha, x)$, for any $\varepsilon_1 > 0$ there exists $\delta_1 > 0$ such that $\rho(\alpha x, 0) < \varepsilon_1$ if $|\alpha| < 1$ and $\rho(x, 0) < \delta$. But this means that the mapping $f : P \times F \rightarrow F$ given by $f(\alpha, x) = \alpha x$ is continuous. ■

The following proposition, a simple corollary of the uniform boundedness principle, is often useful.

PROPOSITION 3. *The following set of conditions is necessary and sufficient for a sequence of bounded linear transformations $\{A_n\}$ mapping the Banach space B into the normed space N to converge pointwise to a bounded linear transformation A_0 .*

- a) the sequence $\{\|A_n\|\}$ is bounded;
- b) $A_n x \rightarrow A_0 x$ for any x in some set X whose linear span is dense in B (i.e., the closure of the set of linear combinations of elements of X coincides with B).

PROOF: The necessity of condition a) is the assertion of Theorem 4, and the necessity of condition b) is obvious. We shall now prove the sufficiency. Let $M = \sup_{n=0,1,\dots} \|A_n\|$ and let $V(X)$ be the linear manifold spanned by the set X . By the linearity of the operators A_n and A_0 and condition b) we have $A_n x \rightarrow A_0 x$ for any $x \in V(X)$. Now choose an element $\xi \notin V(X)$, $\xi \in B$. Let $\varepsilon > 0$ and let the element $x \in V(X)$ be such that $\|x - \xi\| < \frac{\varepsilon}{4M}$.

We write $\|A_n \xi - A_0 \xi\| \leq \|A_n \xi - A_n x\| + \|A_n x - A_0 x\| + \|A_0 x - A_0 \xi\| < \|A_n x - A_0 x\| + \frac{\varepsilon}{2}$. Since $A_n x \rightarrow A_0 x$, there exists an index n_0 such that the inequality $\|A_n x - A_0 x\| < \varepsilon/2$ holds for $n \geq n_0$. For these same indices $n \geq n_0$ we have the inequality

$$\|A_n \xi - A_0 \xi\| < \varepsilon. \quad \blacksquare$$

We remark that this proposition holds in particular for sequence of bounded linear functionals $\{F_n\}$.

2.3. The Bounded Inverse Theorem. The Open Mapping Principle.

Let the transformation A map the Banach space B_1 into the Banach space B_2 . The inverse transformation (inverse mapping), if it exists, will be denoted A^{-1} .

The following lemmas hold.

LEMMA 1. *The transformation A^{-1} inverse to a linear transformation is linear.*

PROOF: Let $Ax_1 = y_1$ and $Ax_2 = y_2$. Then $A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 y_1 + \alpha_2 y_2$ for all $\alpha_1, \alpha_2 \in P$. By definition of the inverse mapping (cf. Sec. 1.1.1) we can write $A^{-1}y_1 = x_1$ and $A^{-1}y_2 = x_2$. Multiplying these equalities by α_1 and α_2 respectively and adding them, we obtain $\alpha_1 A^{-1}y_1 + \alpha_2 A^{-1}y_2 = \alpha_1 x_1 + \alpha_2 x_2$. But by definition of the inverse transformation we have

$$\alpha_1 x_1 + \alpha_2 x_2 = A^{-1}(\alpha_1 y_1 + \alpha_2 y_2).$$

Comparing the last two equalities, we find

$$A^{-1}(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 A^{-1}y_1 + \alpha_2 A^{-1}y_2.$$

This equality holds for any vector of the range of values $R(A)$ of the transformation (the set $\{Ax \in B_2 : x \in D(A)\}$), where $D(A)$ is the domain of definition of the transformation: $\{x \in B_1 : Ax \in B_2\}$. \blacksquare

LEMMA 2. *If A is a continuous linear transformation mapping a Banach space B_1 onto a Banach space B_2 , then the closure of the image (under the mapping A) of any neighborhood of zero in the space B_1 contains a neighborhood of zero in the space B_2 .*

PROOF: Let Σ_0 be an arbitrary neighborhood of zero in the space B_1 . Denote its image under the mapping A by $A\Sigma_0$. One can always exhibit a

neighborhood of zero M_0 in the space B_1 such that $M_0 - M_0 \subset \Sigma_0$. Indeed, as we have already noted, a normed space is in particular a topological vector space, and the existence of the neighborhood M_0 follows from the continuity of addition (since $0 - 0 = 0$).

Further, if $x \in B_1$, then $x/n \rightarrow 0$ as $n \rightarrow \infty$. Hence for sufficiently large n we find $x/n \in M_0$, i.e., $x \in nM_0$. In other words, $B_1 = \bigcup_{n=1}^{\infty} nM_0$, and $B_2 = AB_1 = A\left(\bigcup_{n=1}^{\infty} nM_0\right) = \bigcup_{n=1}^{\infty} nAM_0$. A fortiori $B_2 = \bigcup_{n=1}^{\infty} \overline{nAM_0}$. By the Baire category theorem (Theorem 3 of Sec. 1.3) one of these sets, say $\overline{n_0AM_0}$, is dense in some open set, and therefore contains a nonempty open set. It then follows that the set $\overline{AM_0}$ also contains a nonempty open set. Denote this last open set by V . We have

$$\overline{A\Sigma_0} \supset \overline{AM_0 - AM_0} \supset \overline{AM_0} - \overline{AM_0} \supset V - V.$$

We shall prove that the set $V - V$ is open. Let $x \in V$. Then the set $x - V = \{x - y : y \in V\}$ is open (cf. Property 6 of Sec. 2.1.5). Further $V - V = \bigcup_{a \in V} a - V$, and consequently $V - V$ is open, being the union of open sets. The point 0 belongs to $V - V$, and so this set is a neighborhood of zero. ■

LEMMA 3. *Let A be a continuous linear transformation mapping the Banach space B_1 onto the Banach space B_2 . Then the image of every neighborhood of zero in the space B_1 contains a neighborhood of zero in the space B_2 .*

PROOF: Let $\varepsilon > 0$, $X_\varepsilon = \{x \in B_1 : \rho(x, 0) < \varepsilon\}$, and $Y_\varepsilon = \{y \in B_2 : \rho(y, 0) < \varepsilon\}$.^{*} Choose $\varepsilon_0 > 0$ and let ε_i be an arbitrary sequence of positive numbers such that $\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon_0$. According to Lemma 2 there exists a sequence of positive numbers $\eta_i > 0$, $\eta_i \rightarrow 0$, $i = 0, 1, 2, \dots$, $\eta_0 > \eta_1 > \dots$, such that $\overline{AX_{\varepsilon_i}} \supset Y_{\eta_i}$, $i = 0, 1, 2, \dots$.

Let $y \in Y_{\eta_0}$. We shall find an element $x \in X_{2\varepsilon_0}$ such that $Ax = y$. Indeed, since $\overline{AX_{\varepsilon_0}} \supset Y_{\eta_0}$, there exists $x_0 \in X_{\varepsilon_0}$ such that $\rho(y - Ax_0, 0) = \|y - Ax_0\| < \eta_1$. In exactly the same way, since $y - Ax_0 \in Y_{\eta_1}$, there exists $x_1 \in X_{\varepsilon_1}$ such that $\|y - Ax_0 - Ax_1\| < \eta_2$. Continuing this process indefinitely, we construct vectors x_n such that $x_n \in X_{\varepsilon_n}$ and

$$\left\| y - A\left(\sum_{i=0}^n x_i\right) \right\| < \eta_{n+1}, \quad n = 0, 1, 2, \dots$$

^{*}Here $\rho(x, 0) = \|x\|$ and $\rho(y, 0) = \|y\|$ since the spaces are normed.

We set $z_k = \sum_{i=0}^k x_i$. then $\|z_{k+p} - z_k\| = \rho(z_{k+p} - z_k, 0) = \rho\left(\sum_{i=k+1}^{k+p} x_i, 0\right) \leq \sum_{i=k+1}^{k+p} \varepsilon_i$ for $p > 0$. The sequence $\{z_k\}$ is fundamental, and since B_1 is complete, it converges. Hence we obtain

$$\begin{aligned} x &= \lim_{k \rightarrow \infty} z_k = \lim_{k \rightarrow \infty} \sum_{i=0}^k x_i; \quad \rho(x, 0) = \|x\| \\ &= \lim_{k \rightarrow \infty} \rho(z_k, 0) \leq \lim_{k \rightarrow \infty} \left(\sum_{i=0}^k \varepsilon_i \right) < 2\varepsilon_0. \end{aligned}$$

The transformation A is continuous. Using the continuity of the distance function and this property, we obtain

$$\lim_{n \rightarrow \infty} \rho\left(y - A\left(\sum_{i=0}^n x_i\right), 0\right) = \lim_{n \rightarrow \infty} \left\| y - A\left(\sum_{i=0}^n x_i\right) \right\| \leq \lim_{n \rightarrow \infty} \eta_{n+1} = 0,$$

i.e.,

$$y = A \lim_{n \rightarrow \infty} \sum_{i=0}^n x_i = Ax.$$

Consequently the ball $X_{2\varepsilon_0}$ in B_1 with center at the origin has as its image the set $AX_{2\varepsilon_0}$ containing the ball Y_{η_0} in B_2 with center at the origin. Thus under the mapping A the image of a neighborhood of zero in the space B_1 contains some neighborhood of zero in the space B_2 . ■

LEMMA 4. *Let A be a continuous linear transformation mapping the Banach space B_1 onto the Banach space B_2 . Then the image of every open subset of B_1 is an open subset of B_2 .*

PROOF: Let $\Sigma \subset B_1$ be a nonempty open set, $x \in \Sigma$, and Σ_0 a neighborhood of zero in B_1 such that $x + \Sigma_0 \subset \Sigma$. Let σ_1 be a neighborhood of zero in B_2 such that $A\Sigma_0 \supset \Sigma_1$. Such a neighborhood Σ_1 always exists by Lemma 3. We write the following obvious relations:

$$A\Sigma \supset A(x + \Sigma_0) = Ax + A\Sigma_0 \supset Ax + \Sigma_1.$$

We emphasize that the set $Ax + \Sigma_1$ is an open set—a neighborhood of the point Ax . Thus, since x is an arbitrary point of Σ , i.e., Ax is an arbitrary point of the image $A\Sigma$, the set $A\Sigma$ contains a neighborhood of each of its points. Hence the set $A\Sigma$ is open. ■

A consequence of the lemmas just proved is the following theorem.

THEOREM 7 (Banach's bounded inverse theorem). *Let A be a continuous linear transformation mapping a Banach space B_1 in a one-to-one manner onto a Banach space B_2 . Then the inverse transformation is also linear and continuous.*

PROOF: It follows from Lemma 1 that the transformation A^{-1} is linear. According to Lemma 4 the transformation A takes open sets into open sets. Consequently, the preimage of any open set under the inverse mapping A^{-1} is open, i.e., according to Lemma 4 of Sec. 1.2.4 the mapping A^{-1} is continuous. ■

Looking at the proof of Theorem 7 again, one can verify that all the reasoning is applicable also in the case of a mapping from one F -space onto another. Thus we have also proved the following theorem.

THEOREM 8. *Let A be a continuous linear transformation mapping the F -space F_1 in a one-to-one manner onto the F -space F_2 . Then the inverse operator A^{-1} is also linear and continuous.*

The assertion in Lemma 4 also admits an extension to F -spaces.

THEOREM 8' (Open mapping principle). *Let A be a continuous linear transformation from one F -space onto another. Then the image of each open set is an open set.*

We now return to our study of the concepts of continuity and boundedness of a transformation. In the case of Banach spaces continuity and boundedness of a linear transformation are equivalent concepts (cf. Theorem 1 of Sec. 2.1.1). In the general case of a topological vector space, as we have already said, boundedness of a transformation does not imply continuity. However, if the topological vector space is metrizable, i.e., its topology can be given by a metric, then the following properties are equivalent:

- 1) A is continuous;
- 2) A is bounded;
- 3) the set $\{Ax_n : n = 1, 2, \dots\}$ is bounded if $x_n \rightarrow 0$;
- 4) $Ax_n \rightarrow 0$ if $x_n \rightarrow 0$.

We shall prove only that boundedness and continuity are equivalent for linear transformations between F -spaces.

THEOREM 1'. *Let F_1 and F_2 be two F -spaces and A a linear transformation mapping F_1 into F_2 . Then a necessary and sufficient condition for the transformation A to be continuous is that it be bounded.*

PROOF: If A is a continuous linear transformation and the set Σ is bounded, then, since an F -space is a topological vector space, the set $A\Sigma$ is bounded, so that the transformation A is bounded by Theorem 2 of Sec. 2.2.1.

Now let A be a bounded linear transformation, i.e., let it map bounded sets into bounded sets. We shall prove that it is continuous. We begin by proving that it is continuous at the point 0. Let $x_n \in F_1$ and $\lim_{n \rightarrow \infty} x_n = 0$. Then $\lim_{n \rightarrow \infty} \rho(x_n, 0) = 0$. Choose a sequence of natural numbers k_n such that $\lim_{n \rightarrow \infty} k_n = \infty$, and $\lim_{n \rightarrow \infty} k_n \rho(x_n, 0) = 0$. We have

$$\rho(k_n x_n, 0) = \rho(x_n + x_n + \cdots + x_n, 0) \leq k_n \rho(x_n, 0).$$

Passing to the limit in this inequality as $n \rightarrow \infty$, we find $\lim_{n \rightarrow \infty} \rho(k_n x_n, 0) = 0$, i.e., $\lim_{n \rightarrow \infty} k_n x_n = 0$. Consider the set $B = \{k_n x_n\}$. This set is obviously bounded. Consequently the set $A\{k_n x_n\} = \{k_n A x_n\}$ is bounded, since A is a bounded transformation. Hence $\lim_{n \rightarrow \infty} A x_n = \lim_{n \rightarrow \infty} \frac{1}{k_n} A k_n x_n = 0$, (cf. Sec. 2.1.5).

Thus the transformation A has been proved continuous at zero and by linearity must be continuous everywhere. ■

COROLLARY. Every linear mapping of one F -space into another that takes every sequence converging to zero into a bounded set is continuous.

2.4. Extension of Transformations and Functionals.

The Hahn-Banach Extension Theorem

If a transformation or functional is defined on only a linear manifold V' in a vector space V , the question naturally arises whether it can be extended to the entire space without losing certain of its properties. In other words, it is required to construct a new transformation or functional possessing certain properties and coinciding with the original on V' .

For linear transformations this question is easily decided if the original transformation is defined on a linear manifold that is a dense subset of the whole space.

We have the following theorem.

THEOREM 9. A bounded linear transformation A_0 defined on a dense linear manifold V' of a normed vector space N with values in a Banach space B can be extended to the entire space N without increasing its norm. To be specific, one can define a transformation A on all of N such that $Ax = A_0x$ for $x \in V'$, and $\|A\|_N = \|A_0\|_{V'}$.

PROOF: Let $x \in N$ but $x \notin V'$. Since $\bar{V}' = N$, there exists a sequence $\{x_n\}$ belonging to V' such that $x_n \rightarrow x$ as $n \rightarrow \infty$ (cf. Lemma 3 of Sec. 2.1.4).

Consequently the sequence $\{x_n\}$ is fundamental. Then $\|A_0 x_n - A_0 x_m\| \leq \|A_0\|_{V'} \|x_n - x_m\| \rightarrow 0$ as $n, m \rightarrow \infty$. It follows from this that the sequence $\{A_0 x_n\}$ is fundamental and by the completeness of the Banach space B it converges to some element $y \in B$. We define $Ax = y = \lim_{n \rightarrow \infty} A_0 x_n$. If $\{x'_n\}$ is another sequence of V' converging to the element $x \in N$, then $\|A_0 x_n - A_0 x'_n\| \leq \|A_0\|_{V'} \|x_n - x'_n\| \rightarrow 0$ as $n \rightarrow \infty$, i.e., $A_0 x_n - A_0 x'_n \rightarrow 0$ as $n \rightarrow \infty$, and the transformation A is unambiguously defined on the elements of N not belonging to V' .

If $x \in V'$, we set $x = x_n$ for all n and

$$Ax = \lim_{n \rightarrow \infty} A_0 x_n = A_0 x.$$

Consequently the transformation A is unambiguously defined in this case also.

The transformation A so constructed is linear because of the properties of limits. It is also bounded, since $\|A_0 x_n\| \leq \|A_0\| \|x_n\|$, and, passing to the limit in this inequality, we find that $\|Ax\| \leq \|A_0\|_{V'} \|x\|$, i.e., $\|A\|_N \leq \|A_0\|_{V'}$. We remark that in the extension of the transformation the norm cannot decrease, so that $\|A\|_N = \|A_0\|_{V'}$. ■

The extension procedure just exhibited is called extension by continuity. The theory of extensions of transformations constitutes an independent and interesting part of functional analysis.

If a continuous linear functional is defined (cf. Definition 11 of Sec. 2.1.2), it can be extended without increasing its norm, even when its original domain is not dense in the space. The corresponding theorem plays an important role in analysis and is called the *Hahn-Banach extension theorem*.

THEOREM 10. (Hahn-Banach extension theorem). Suppose a gauge function* is defined on a real vector space V , i.e., a real-valued function such that

$$p(x_1 + x_2) \leq p(x_1) + p(x_2), \quad p(\lambda x) = \lambda p(x), \quad x_1, x_2 \in V, \quad \lambda \geq 0.$$

Let $f(x)$ be a real-valued linear functional defined on a linear manifold $V' \subset V$ and such that

$$f(x) \leq p(x), \quad x \in V'.$$

Then there exists a real-valued linear functional F defined on all of V such that:

$$1) F(x) = f(x) \text{ for } x \in V';$$

*Cf. Definition 18 of Sec. 2.1.4.

2) $F(x) \leq p(x)$ for $x \in V$.

If V is a real normed vector space and f is a bounded functional on $V' \subset V$, then $p(x)$ can be defined as $p(x) = \|x\| \|f\|_{V'}$, for all $x \in V$, and then we have the equality

2') $\|F\|_V = \|f\|_{V'}$,

i.e., the functional f can be extended to a continuous functional F on V without changing its norm.

PROOF: We shall show that if $V' \neq V$, then the functional f can be extended from V' to some larger linear manifold V_0 .

Suppose the element x belongs to V but not to V' . We consider the set $(V'; x) = V_0$ of elements of the form $tx + x_0$, $x_0 \in V'$, t any real number. It is obvious that V_0 is a linear manifold and easy to verify that each element of V_0 has a unique representation in the form $tx + x_0$. We denote by f_1 the desired extension of the functional f to V_0 . We set

$$f_1(tx + x_0) = tf_1(x) + f(x_0) = tc + f(x_0),$$

where we must choose the number $c = f_1(x)$ so that the inequality

$$f_1(tx + x_0) \leq p(tx + x_0)$$

holds in V_0 . To do this we first consider the case $t > 0$. Then this inequality is equivalent to the following:

$$f\left(\frac{x_0}{t}\right) + c \leq p\left(\frac{x_0}{t} + x\right),$$

or

$$c \leq p\left(\frac{x_0}{t} + x\right) - f\left(\frac{x_0}{t}\right),$$

where $c = f_1(x)$, $x \in V$.

If $t < 0$ the condition

$$f\left(\frac{x_0}{t}\right) + c \geq -p\left(-\frac{x_0}{t} - x\right),$$

arises analogously, i.e.,

$$c \geq -p\left(-\frac{x_0}{t} - x\right) - f\left(\frac{x_0}{t}\right).$$

We remark that there always exists a number c satisfying these two relations. Indeed, let x' and x'' be arbitrary elements of V' , then

$$\begin{aligned} f(x'') - f(x') &\leq p(x'' - x') = p((x'' + x) - (x' + x)) \\ &\leq p(x'' + x) + p(-x' - x), \end{aligned}$$

i.e.,

$$-f(x'') + p(x'' + x) \geq -f(x') - p(-x' - x).$$

Let

$$c'' = \inf_{x'' \in V'} (-f(x'') + p(x'' + x)), \quad c' = \sup_{x' \in V'} (-f(x') - p(x' - x)).$$

Since x'' and x' are arbitrary, it follows from the preceding inequalities that $c'' \geq c'$. Choosing c so that $c'' \geq c \geq c'$, we define a functional f' on V_0 by the rule

$$f_1(tx + x_0) = tc + f(x_0).$$

Under such a choice of the number c on V_0 the inequality

$$f_1(tx + x_0) \leq p(tx + x_0)$$

holds. Thus the desired extension of the functional f from V' to V_0 is obtained. If we can choose a countable system of elements x_1, x_2, x_3, \dots that generate V ,* then the functional F can be constructed by induction, considering the increasing sequence of linear manifolds $V_1 = (V'; x_1)$, $V_2 = (V_1, x_2)$, \dots ; each time V_{k+1} is the smallest linear manifold containing the linear manifold V_k and the element x_k (i.e., the intersection of all such linear manifolds). Every element $x \in V$ belongs to some V_k , so that the function will be extended to the entire space V .

In the general case, i.e., when no countable set of generators $\{x_i\}_{i=1}^{\infty}$ of the space V exists, we use the following reasoning. Let $\Phi_{V'}$ be the set of all extensions of the functional f satisfying the inequality $f(x) \leq p(x)$. As shown above, such extensions exist. On this set we introduce an order relation, namely $f' < f''$ for $f', f'' \in \Phi_{V'}$ if the linear manifold V'_0 on which f' is defined is contained in the linear manifold V''_0 on which f'' is defined and $f'(x) = f''(x)$ for $x \in V'_0$. It is easy to verify that all the properties defining an order relation are satisfied, and the set $\Phi_{V'}$ is thereby partially ordered.

Now let $\{f_\alpha\}$ be an arbitrary subset of $\Phi_{V'}$ linearly ordered by this relation. This subset has an upper bound, which is the functional \hat{f} defined on the linear manifold $\hat{V} = \bigcup_{\alpha} V_\alpha$, where V_α is the domain of definition of f_α and $\hat{f}(x) = f_{\alpha_0}(x)$, if $x \in \hat{V}$ is an element of V_{α_0} . It is obvious that \hat{f} is a linear functional, $\hat{f} \in \Phi_{V'}$.

*That is, the vector space V is spanned by the set $\{x_i\}_{i=1}^{\infty}$, which means that every element of V is a linear combination of the elements of this set.

Thus all the hypotheses of Zorn's lemma are satisfied, and $\Phi_{V'}$ has a maximal element F . This functional is defined on all of V , since otherwise it could be extended, and F would not be a maximal element of $\Phi_{V'}$.

If V is a normed vector space, we can take as $p(x)$ the functional $\|f\|_{V'}\|x\|_V = p(x)$, $x \in V$, throughout the proof. It is clear that this functional has the properties stated in the theorem. ■

EXAMPLES

1. If the real-valued function $p(x)$ over a vector space V is a gauge function, i.e., has the properties

- a) $p(x+y) \leq p(x) + p(y)$ for any $x, y \in V$,
- b) $p(\alpha x) = \alpha p(x)$ for any $x \in V$ and $\alpha > 0$,

then for any $\alpha \in \mathbf{R}^1$ we shall have $p(\alpha x) \geq \alpha p(x)$. Indeed, for $\alpha > 0$ this is obvious. For $\alpha = 0$ it follows from the relations $p(\alpha \cdot 0) = p(0) = \alpha p(0)$ for $\alpha > 0$, i.e., $p(0) = 0$. Now let $\alpha < 0$. We have $p(\alpha x) + p(|\alpha|x) = p(\alpha x) + |\alpha|p(x)$. But for any element $z \in V$ we have $p(z + (-z)) = p(0) \leq p(z) + p(-z)$. Therefore $p(\alpha x) + |\alpha|p(x) \geq 0$, i.e., $p(\alpha x) \geq -|\alpha|p(x) = \alpha p(x)$.

2. The functionals $p(x) = \|f\| \cdot \|x\|$, where f is a linear functional on a normed vector space, and $p_m(x) = \sup_n |x_n|$ over the vector space m of bounded sequences are examples of both gauge functions and seminorms. A linear functional is also an example of a gauge function and a seminorm (cf. Definition 18 of Sec. 2.1.4).

3. If N is a normed space and $x_0 \in N$, there exists a linear functional $F(x)$ defined on all of N and such that $F(x_0) = \|x_0\|$, $|F(x)| \leq \|x\|$, $x \in N$. Indeed, let $x_0 \neq 0$, and set $\{tx_0\} = N'$ for t in the field P of real numbers. Then N' is a linear manifold. We define a functional $f(x)$ on N' by the rule $f(x) = t\|x_0\|$. It is obvious that $f(x_0) = \|x_0\|$ and $|f(x)| = |t| \cdot \|x_0\| = \|x\|$, i.e., $\|f\| = 1$. Extending the functional $f(x)$ to all of the space without increasing its norm, we obtain the desired result. We remark that $\|F\| = 1$. If $x_0 = 0$, we set $F = 0$.

4. For sequences $\{\alpha_n\} \in l^2$, and $\{\beta_n\} \in l^2$, we have by the Cauchy-Bunyakovskii inequality $(\sum |\alpha_n \beta_n|)^2 \leq \sum |\alpha_n|^2 \sum |\beta_n|^2$, i.e., the sequence $\{\alpha_n \beta_n\}$ belongs to l^1 (cf. the examples of Sec. 1.2.1).

The converse assertion is also true. To be specific, if $\sum |\alpha_n \beta_n| < \infty$ for any sequence such that $\sum |\alpha_n|^2 < \infty$, then $\sum |\beta_n|^2 < \infty$. Indeed, let $\gamma_k = (\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_k, 0, 0, \dots)$, where the bar denotes complex conjugation. It is clear that $\gamma_k \in l^2$, $k = 1, 2, \dots$. Let $f = (\alpha_1, \alpha_2, \dots)$ be any vector of l^2 . Then $F_{\gamma_k}(f) = \sum_{j=1}^k \alpha_j \beta_j \rightarrow \sum_{j=1}^{\infty} \alpha_j \beta_j$ as $k \rightarrow \infty$. Therefore for each

$f \in l^2$ the sequence F_{γ_k} converges to some element $F_\gamma(f) = \sum_{j=1}^{\infty} \alpha_j \beta_j$. By the uniform boundedness principle, the norms of the functionals F_{γ_k} are uniformly bounded: $\|F_{\gamma_k}\| \leq C$ for all k . But $\|F_{\gamma_k}\| = \left(\sum_{j=1}^k |\beta_j|^2 \right)^{1/2}$, so that $\{\beta_n\} \in l^2$.

5. Let B be a Banach space, E the identity operator on B , and A a bounded linear transformation mapping B into itself such that $\|A\| \leq q < 1$. Then the operator $(E - A)^{-1}$ exists, is bounded, and is representable in the form $(E - A)^{-1} = \sum_{k=0}^{\infty} A^k$, where this last series converges in the space of linear operators ($B \rightarrow B$), i.e., the sequence of partial sums of the series $S_n = \sum_{k=0}^n A^k$ converges uniformly. Indeed, since B is complete, to prove that the sequence S_n converges it suffices to prove that it is fundamental. But for any integer $p > 0$

$$\|S_{n+p} - S_n\| = \|A^{n+1} + A^{n+2} + \dots + A^{n+p}\| \leq \|A\|^{n+1} + \dots + \|A\|^{n+p}.$$

This follows from the fact that if the operators B and C belong to the space $(N_1 \rightarrow N_1)$, then $\|B + C\| \leq \|B\| + \|C\|$ and $\|BCx\| \leq \|B\|\|C\|\|x\|$, i.e., $\|BC\| \leq \|B\|\|C\|$ or $\|A^k\| \leq \|A\|^k$ (provided, of course, the inclusion $R(C) \subset D(B)$ holds, i.e., the range of values of the operator C is contained in the domain of definition of the operator B). Thus $\|S_{n+p} - S_n\| \leq q^{n+1}/(1 - q) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, as we have already said, by the completeness of B , $\sum_{k=0}^{\infty} A^k$ is a bounded linear operator. Further, for any n we have

$$(E - A) \sum_{k=0}^n A^k = \sum_{k=0}^n A^k (E - A) = E - A^{n+1}.$$

We now pass to the limit as $n \rightarrow \infty$. Since $\|A^{n+1}\| \leq \|A\|^{n+1} \rightarrow 0$ as $n \rightarrow \infty$, we have $(E - A) \sum_{k=0}^{\infty} A^k = \sum_{k=0}^{\infty} A^k (E - A) = E$. From this, as is easy to see, $(E - A)^{-1} = \sum_{k=0}^{\infty} A^k$, which was to be shown. (The existence and boundedness of the operator $(E - A)^{-1}$ also follow from the next example.)

6. Consider the space $(B_1 \rightarrow B_2)$. Let $U(B_1 \rightarrow B_2)$ be the subset of $(B_1 \rightarrow B_2)$ consisting of the transformations mapping B_1 into B_2 and having a bounded inverse. This set is open in $(B_1 \rightarrow B_2)$. For let $A_0 \in U(B_1 \rightarrow B_2)$ and let A_δ be an arbitrary transformation of $(B_1 \rightarrow B_2)$ such that $\|A_\delta\| < \frac{1}{\|A_0^{-1}\|}$. Then the operator $(A_0 + A_\delta)^{-1}$ exists and is bounded,

i.e., $A_0 + A_\delta \in U(B_1 \rightarrow B_2)$.^{*} In fact let $y \in B_2$ and $Ax = A_0^{-1}y - A_0^{-1}A_\delta x$. Since $\|A_\delta\| < \|A_0^{-1}\|^{-1}$, the mapping A is a contraction:

$$\rho(Ax', Ax'') = \|A_0^{-1}A_\delta(x' - x'')\| < q\rho(x', x''), \quad q < 1, \quad x', x'' \in B_1,$$

where the distance function is induced by the norm of the corresponding space. Since the space B_1 is complete, there exists a unique fixed point x for the mapping A , i.e., $x = Ax = A_0^{-1}y - A_0^{-1}A_\delta x$. Hence $A_0x + A_\delta x = y$. If there exists x' such that $A_0x' + A_\delta x' = y$, then x' is also a fixed point of the mapping, so that $x' = x$. Thus for every $y \in B_2$ the equation $A_0x + A_\delta x = y$ has a unique solution. Hence by the definition of the inverse of a mapping, the transformation $(A_0 + A_\delta)^{-1}$ is defined on all of B_2 and establishes a one-to-one correspondence between B_2 and B_1 . By Theorem 7 on the inverse transformation it is bounded, which was to be proved.

2.5. Different Topologies, Different Types of Convergence.

The General Forms of Functionals in Particular Spaces.

Several kinds of convergence can be defined in the space of transformations. In this connection we give a series of definitions.

DEFINITION 5. Convergence in the norm of the space of bounded linear transformations $(N_1 \rightarrow N_2)$ is called *uniform convergence*. In other words if a sequence of linear transformations A_n converges in norm to a transformation A as $n \rightarrow \infty$, i.e., $\|A_n - A\| \rightarrow 0$ as $n \rightarrow \infty$, we say that $\{A_n\}$ converges to A uniformly as $n \rightarrow \infty$. We denote this kind of convergence by writing $A_n \Rightarrow A$ as $n \rightarrow \infty$.

In the same space of transformations $(N_1 \rightarrow N_2)$ we can also introduce another kind of convergence, called *pointwise convergence*.

DEFINITION 6. We shall say that a sequence of transformations A_n converges *pointwise* to the transformation A in the space $(N_1 \rightarrow N_2)$ as $n \rightarrow \infty$ if for each $x \in N_1$ the relation $\lim_{n \rightarrow \infty} A_n x = Ax$ holds. We denote pointwise convergence in the space $(N_1 \rightarrow N_2)$ of bounded linear transformations by writing $A_n \rightarrow A$ as $n \rightarrow \infty$.

It is easy to verify that uniform convergence of a sequence $\{A_n\} \in (N_1 \rightarrow N_2)$ implies pointwise convergence, but the converse in general is not true. Indeed, let $N_1 = N_2 = l^2$ and let A_n denote projection on the linear manifold in l^2 generated by the elements $e_1 = (1, 0, \dots), e_2 =$

^{*}Consequently the set $U(B_1 \rightarrow B_2)$ contains a neighborhood of an arbitrary element $A_0 \in U(B_1 \rightarrow B_2)$, which means the set $U(B_1 \rightarrow B_2)$ is open.

$(0, 1, 0, \dots), \dots, e_n = (0, \dots, 0, 1, 0, \dots)$, where the 1 in the last vector follows $n - 1$ zeros. Then for any $\xi \in l^2$

$$A_n \xi = \sum_{i=1}^n (\xi, e_i) e_i \rightarrow \sum_{i=1}^{\infty} (\xi, e_i) e_i = \xi, \quad (\xi, e_i) = \xi_i, \quad \xi = (\xi_1, \xi_2, \dots),$$

and therefore $A_n \rightarrow E$, pointwise, where E is the identity transformation. On the other hand $\|A_n e_{n+1} - A_{n+p} e_{n+1}\| = 1$ (where the norm of the element

is taken in the space l^2 , i.e., $\|\xi\| = \rho(\xi, 0) = \sqrt{\sum_{i=1}^{\infty} |\xi_i|^2}$, $\xi = (\xi_1, \xi_2, \dots)$).

Since the norm of the vector e_{n+1} is 1, we have $\|A_n - A_{n+p}\| = \sup_{\|x\|=1} \|A_n x - A_{n+p} x\| \geq 1$, so that the convergence of A_n is not uniform.

We recall that we defined the dual space to a vector space earlier as the set of continuous linear functionals mapping the space V_T into the field of scalars. In what follows it is convenient to introduce the notation V_T^* for the dual space of V_T .

The dual space is a particular case of a space of transformations, and so in particular the convergence defined above can be introduced into it.

Therefore we can introduce two topologies on the space—the strong topology and the weak topology. The former corresponds to uniform convergence in the space of functionals dual to a normed space, the latter to pointwise convergence. (We remark that in the case of the dual space pointwise convergence is called weak convergence, as will be emphasized again below).

We begin by considering the case when the initial space is normable. We give the following definitions.

DEFINITION 7. Let N be a normable vector space and N^* its dual space. The *strong topology* in the space N^* is the topology corresponding to the norm on N^* (if $f \in N^*$, then $\|f\|_{N^*} = \sup_{\|x\|_{N^*} \leq 1} |f(x)|$). In other words

a neighborhood of zero in the space N^* is taken to be the set of functionals $\{f\}$ satisfying the condition $\|f\| < \varepsilon$; or, in other words, a neighborhood of zero in the space N^* is the set of functionals $\{f\}$ for which $|f(x)| < \varepsilon$ when x belongs to the closed unit ball $K : \|x\| \leq 1$. Taking all possible ε , we obtain a neighborhood basis at zero: $\Sigma_{\varepsilon, K}^* = \{f : |f(x)| < \varepsilon, x \in K\}$.

We emphasize that the convergence defined by this topology in the dual space, according to Definition 5, is called uniform convergence. In the case of the dual space to a normed space the name *strong convergence* is also frequently used.

Now, as before, let N be a normed vector space and N^* its dual space. Let B_n be an arbitrary subset of the space N consisting of n elements of the space N , ($n < \infty$), $B_n = \{x_i\}_{i=1}^n$. We take the neighborhoods of zero in the space N^* to be sets of the form $\Sigma_{\epsilon, B_n}^* = \{f : |f(x)| < \epsilon, x_i \in B_n\}$, where ϵ is an arbitrary positive number. It is clear that we have thereby defined a neighborhood basis at zero (cf. Definition 3 of Sec. 1.4).

We now give the following definition.

DEFINITION 8. Let N be a normed vector space and N^* its dual space. The *weak-star topology* in the space N^* is the topology given by the following neighborhood basis of zero in the space N^* : $\Sigma_{\epsilon, B_n}^* = \{f : |f(x_i)| < \epsilon, x_i \in B_n\}$, where ϵ is an arbitrary positive number and B_n an arbitrary n -element subset of N , $n < \infty$.

The weak-star topology introduced above defines a kind of convergence in the space N^* known as weak-star convergence. To be specific, a sequence of functionals $f_m \in N^*$ is said to be *weak-star convergent* to a functional $f \in N^*$ as $m \rightarrow \infty$ if for each element $x \in N$ the relation $f_m(x) \rightarrow f(x)$ as $m \rightarrow \infty$ holds. In other words, weak-star convergence is convergence at each individual element. This kind of convergence was called pointwise convergence in the space of linear transformations. Because of the inequality $|f_m(x) - f(x)| \leq \|f_m - f\| \|x\|$, strong convergence of functionals implies their weak-star convergence. The converse in general is not true. Let us show that the weak-star topology does indeed define weak-star convergence. For simplicity assume $f = 0$. Suppose for any $x \in N$ we have $f_m(x) \rightarrow 0$ as $m \rightarrow \infty$. Then for any neighborhood of zero $\Sigma_{\epsilon, B_n}^* = \{f : |f(x_i)| < \epsilon, x_i \in B_n\}$ there exists M such that $f_k \in \Sigma_{\epsilon, B_n}^*$ for all $k \geq M$. Indeed, it suffices to take M_i such that $|f_k(x_i)| < \epsilon$ for $k \geq M_i$ and then set $M = \max_{1 \leq i \leq n} M_i$.

The converse is obvious.

Just as in the space of functionals N^* a topology can be introduced in the original space N in various ways. We shall give details of only two ways of introducing the topology. In this connection we give the following definitions.

DEFINITION 7'. Let N be a normable vector space. The *strong topology* in the space N is the topology corresponding to the norm introduced in N . In other words a neighborhood of zero in the space N is taken as the set of elements $\{x\}$ satisfying the inequality $\|x\| < \epsilon$. Taking all possible ϵ , we obtain a neighborhood basis at zero: $\Sigma_\epsilon = \{x : \|x\| < \epsilon\}$.

The convergence defined by this topology in the space N is called strong convergence in the original space N .

Now, as before, let N be a normed vector space and N^* its dual space. Let B_n^* be a finite subset of N^* consisting of n elements of the space N^* , $n < \infty$, $B_n^* = \{f_i\}_{i=1}^n$. We take the neighborhoods of zero in the space N to be the sets of the form $\Sigma_{\varepsilon, B_n^*} = \{x : |f_i(x)| < \varepsilon, f_i \in B_n^*\}$, where ε is an arbitrary positive number.

DEFINITION 8'. Let N be a normed vector space and N^* the space dual to it. The *weak topology* in the space N is the topology given by the following system of neighborhoods of zero in the space N : $\Sigma_{\varepsilon, B_n^*} = \{x : |f_i(x)| < \varepsilon, f_i \in B_n^*\}$, where ε is an arbitrary positive number and B_n^* is an arbitrary n -element subset of the space N^* , $n < \infty$.

The weak topology determines a kind of convergence in the space N called weak convergence. To be specific, a sequence of elements $x_m \in N$ converges weakly to the element $x \in N$ as $m \rightarrow \infty$ if for any element $f \in N^*$ the relation $f(x_m) \rightarrow f(x)$ as $m \rightarrow \infty$ holds. The proof of this assertion is an exact repetition of the the proof given in the case of weak-star convergence in the space N^* . In exactly the same way we conclude that strong convergence of a sequence in the space N implies its weak convergence. The converse in general is not true.

We remark that the strong, weak-star, and weak topologies have been defined above only in the case when the initial space is normed.

Definitions 7, 8, 7', and 8' remain valid also in the case when the initial space is a topological vector space V_T . In this case the unit ball in the definitions given above must be replaced by a bounded set in V_T . Of course the initial topology in the space V_T is now defined by some system of open sets and not by a norm in the space. We remark further that the weak topology in the space V_T no longer necessarily satisfies the Hausdorff separation axiom.

The continuous linear functionals on a normed vector space N themselves form a normed vector space N^* . Therefore one can construct the space N^{**} dual to N^* , etc. The space N^{**} is called the *second dual space*.

Every element x_0 of N defines a linear functional on N^* . In fact, we set $F_{x_0}(f) = f(x_0)$, where x_0 is a fixed element of N and f ranges over all of N^* . It is obvious that when this is done $F_{x_0}(f)$ is a functional on N^* . Since we also have

$$F_{x_0}(\alpha f_1 + \beta f_2) = \alpha f_1(x_0) + \beta f_2(x_0) = \alpha F_{x_0}(f_1) + \beta F_{x_0}(f_2),$$

it follows that this functional is linear. The functional $F(f)$ is often written as follows: (F, f) .

Furthermore, every such functional is continuous on N^* . Indeed, we have $|F_{x_0}(f)| = |f(x_0)| \leq \|x_0\| \|f\|$, so that $\|F_{x_0}\| \leq \|x_0\|$, i.e., F_{x_0} is bounded, hence continuous.

Thus we have obtained a mapping of the space N onto a linear manifold in the space N^{**} . Such a mapping of the space N into N^{**} is called the *natural mapping*. In fact this mapping is one-to-one. Indeed, as follows from the Hahn-Banach theorem, for any two points x' and x'' of the space N there exists a functional $f(x)$ such that $f(x') \neq f(x'')$ (cf. Example 3 of Sec. 2.2.4), and therefore $F_{x'}$ and $F_{x''}$ are different functionals on N^* .

The natural mapping constructed above is an isomorphism, i.e., it follows from the relations $x \leftrightarrow F_x$ and $y \leftrightarrow F_y$ that $x + y \leftrightarrow F_x + F_y$ and $\lambda x \leftrightarrow \lambda F_x$, $\lambda \in P$. This follows from the relation $F_x(f) = f(x)$ and the linearity of the functional f . This mapping is also an isometry, i.e., the relation $x \leftrightarrow F_x$ implies that $\|x\| = \|F_x\|$. Indeed it follows from the Hahn-Banach theorem (cf. Example 3 of Sec. 2.2.4), that for any $x_0 \in N$, $x_0 \neq 0$, there exists a linear functional f such that $\|f\| = 1$ and $f(x_0) = \|x_0\|$. Therefore $|F_{x_0}(f)| = |f(x_0)| = \|x_0\| = \|f\| \|x_0\|$. Since we always have $\|F_{x_0}\| \leq \|x_0\|$, it follows that $\|F_{x_0}\| = \|x_0\|$, as required. In many interesting cases it happens that there is a one-to-one correspondence between the elements of the original space N and those of the second dual N^{**} that preserves the vector-space operations (an isomorphism) and the distance (an isometry).

DEFINITION 9. If the natural mapping of a normed vector space N maps it onto all of N^{**} , the space N is called *reflexive*. In this case the spaces N and N^{**} can be identified: $N = N^{**}$.

Let us consider in more detail the concepts of weak and strong convergence. Let N be a normed vector space and $\{f_n\}$ a sequence of linear functionals in the dual space N^* . We have noted that if the sequence $\{f_n\}$ is weak-star convergent to a functional $f_0 \in N^*$, then $f_n(x) \rightarrow f_0(x)$ for any $x \in N$, i.e., weak-star convergence of functionals coincides with pointwise convergence.

In terms of weak-star convergence Theorem 5 and Proposition 3 of Sec. 2.2.2 can be stated in a form applicable to functionals as follows:

THEOREM 5'. Let B be a Banach space and B^* its dual (i.e., the space of bounded linear functionals on the space B mapping B into the scalar field). Then the space B^* is complete in the sense of weak-star convergence.

In other words, if a sequence of functionals $\{f_n\}$ is fundamental at every point $x \in B$, there exists a linear functional f such that

$$f_n(x) \rightarrow f(x)$$

for any $x \in B$.

PROPOSITION 3'. The following set of conditions is necessary and sufficient for a sequence of bounded linear functionals $\{f_n\}$ mapping a Banach

space B into the scalar field P to be weak-star convergent to a functional f_0 .

a) The sequence $\{\|f_n\|\}$ is bounded;

b) $f_n(x) \rightarrow f_0(x)$ for all x in any set X whose linear combinations are everywhere dense in B (i.e., the closure of the set of linear combinations coincides with B).

PROPOSITION 3". The following set of conditions is necessary and sufficient for a sequence $\{x_n\} \in B$ to converge weakly to $x_0 \in B$.

a) The sequence $\{\|x_n\|\}$ is bounded;

b) $f(x_n) \rightarrow f(x_0)$ for any f in some set F of linear functionals whose linear combinations are everywhere dense in B^* .

This proposition is a special case of Proposition 3'. Indeed, weak convergence of $\{x_n\} \in B$ to an element $x_0 \in B$ is equivalent to weak-star convergence of the same sequence regarded as a sequence of linear functionals on B^* to the element x_0 regarded as a linear functional on B^* .

We shall prove two more propositions connected with the concept of weak convergence.

PROPOSITION 4. Let the continuous linear transformation A map a normed space N_1 into a normed space N_2 . If the sequence $\{x_n\} \subset N_1$ converges weakly to $x_0 \in N_1$, then the sequence $\{Ax_n\} \subset N_2$ converges weakly to $Ax_0 \in N_2$.

PROOF: Consider an arbitrary linear functional $F \in N_2^*$. Then we have $F(Ax_n) = f(x_n)$, where $f \in N_1^*$, and $F(Ax_0) = f(x_0)$. Since x_n converges weakly to x_0 , we have $f(x_n) \rightarrow f(x_0)$, i.e., $F(Ax_n) \rightarrow F(Ax_0)$, and since F is an arbitrary functional of N_2^* , it follows that Ax_n converges weakly to Ax_0 . Consequently one can say that any continuous linear transformation is also weakly continuous. ■

PROPOSITION 5. A weakly convergent sequence $\{x_n\}$ of elements of a normed space N is bounded, i.e., the norms of the elements of the sequence are uniformly bounded.

PROOF: Indeed, the elements x_n , $n = 1, 2, \dots$, can be regarded as elements of N^{**} . Then weak convergence of the sequence $\{x_n\}$ to an element $x \in N$ means that the sequence of functionals $\{x_n\} \subset N^{**}$ converges to the functional $x \in N^{**}$ at each element $f \in N^*$. But then by Theorem 4 of Sec. 2.2.2 the sequence $\{\|x_n\|\}$ is uniformly bounded. ■

2.5.1. Weak Convergence in the Finite-Dimensional Space \mathbf{R}^n .

We shall show that in a finite-dimensional space weak convergence coincides with strong convergence. Indeed, let $\{x^{(k)}\}$ be a sequence in \mathbf{R}^n that converges weakly to an element x . Let e_1, e_2, \dots, e_n be an orthonormal basis of \mathbf{R}^n . Then we can write

$$\begin{aligned}x^{(k)} &= x_1^{(k)} e_1 + x_2^{(k)} e_2 + \cdots + x_n^{(k)} e_n, \\x &= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.\end{aligned}$$

Since the quantities $x_i^{(k)} = (x^{(k)}, e_i)$, $i = 1, \dots, n$, are continuous linear functionals on \mathbf{R}^n , it follows that $x_1^{(k)} = (x^{(k)}, e_1) \rightarrow (x, e_1) = x_1, \dots, x_n^{(k)} = (x^{(k)}, e_n) \rightarrow x_n$, i.e., the sequence $\{x^{(k)}\}$ converges coordinatewise to x . But then

$$\rho(x^{(k)}, x) = \left(\sum_{i=1}^n |x_i^{(k)} - x_i|^2 \right)^{1/2} \rightarrow 0, \quad k \rightarrow \infty,$$

i.e., $\{x^{(k)}\}$ converges strongly to x . Since strong convergence implies weak convergence, the equivalence of the two notions of convergence in \mathbf{R}^n is proved.

2.5.2. Weak Convergence in l^1 .

We shall show that weak convergence implies coordinatewise convergence in the space l^1 .

Indeed, if $e_1 = (1, 0, 0, \dots), e_2 = (0, 1, 0, \dots), \dots$ is a basis of the space l^1 and the sequence $\{x^{(k)}\}$ converges weakly to an element x , then we can write $x^{(k)} = \sum_{i=1}^{\infty} x_i^{(k)} e_i$, and

$$x = \sum_{i=1}^{\infty} x_i e_i, \quad \text{where } x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots) \text{ and } x = (x_1, x_2, \dots),$$

and the following relations hold:

$$\begin{aligned}x_1^{(k)} &= \sum_{i=1}^{\infty} x_i^{(k)} e_1^i \rightarrow \sum_{i=1}^{\infty} x_i e_1^i = x_1, \\x_2^{(k)} &= \sum_{i=1}^{\infty} x_i^{(k)} e_2^i \rightarrow \sum_{i=1}^{\infty} x_i e_2^i = x_2, \dots, k \rightarrow \infty,\end{aligned}$$

where e_j^i is the i th coordinate of the vector e_j , that is,

$$e_j^i = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The relations just written follow from the weak convergence of the vector $x^{(k)}$ to the vector x . Indeed, for example, the expression $x_1^{(k)} = f(x^{(k)}) = \sum_{i=1}^{\infty} x_i^{(k)} e_i^1$ (here $e_1^1 = 1$ for $i = 1$ and $e_1^1 = 0$ for $i > 1$) is obviously a continuous linear functional. Hence weak convergence in l^1 implies coordinatewise convergence. Since $\|x^{(k)} - x\|_{l^1} = \sum_{i=1}^{\infty} |x_i^{(k)} - x_i|$, it might appear that weak convergence on l^1 actually coincides with strong convergence.

We now turn to the study of the general forms of functionals in particular spaces and to the construction of the dual spaces.

2.5.3. The Finite-Dimensional Space \mathbf{R}^n .

We shall find the general form of a linear functional on \mathbf{R}^n and also the space dual to \mathbf{R}^n .

The inner product of a vector x in \mathbf{R}^n with a fixed vector ξ , as x ranges over the entire space, is obviously a continuous linear functional

$$f(x) = (x, \xi) = \sum_{i=1}^n x_i \xi_i,$$

where x_i and ξ_i are the coordinates of the vectors x and ξ respectively in a basis $\{e_i\}$, $i = 1, \dots, n$. The linearity of the functional $f(x)$ is verified immediately, and its continuity follows from the Cauchy-Bunyakovskii inequality:

$$|(x, \xi)|^2 \leq \|x\|^2 \|\xi\|^2, \text{ where } \|x\|^2 = \sum_{i=1}^n |x_i|^2, \|\xi\|^2 = \sum_{i=1}^n |\xi_i|^2 \text{ (cf. the examples of Sec. 1.2.1).}$$

We shall now establish the converse result. We shall show that every continuous linear functional $f(x)$ on \mathbf{R}^n has the form of the inner product written above with some vector ξ that is uniquely determined by the functional f . Let $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, \dots , $e_n = (0, 0, \dots, 1)$

be a basis in \mathbf{R}^n . For any vector $x \in \mathbf{R}^n$ we have the equality $x = \sum_{i=1}^n x_i e_i$, $x = (x_1, \dots, x_n)$. Consequently for any linear functional f we have

$$f(x) = \sum_{i=1}^n x_i f(e_i) = \sum_{i=1}^n x_i \xi_i = (x, \xi), \quad \xi_i = f(e_i), \quad i = 1, \dots, n.$$

We shall show that the vector ξ is uniquely determined by the functional f . Indeed, suppose $f(x) = (x, \xi)$ and $x = e_i$. Then $f(e_i) = (e_i, \xi) = \xi_i$, i.e., we find that ξ_i is necessary $f(e_i)$.

It follows from the inequality $|f(x)| = |(x, \xi)| \leq \|x\| \|\xi\| \leq M \|x\|$ that the functional $f(x)$ is bounded and, moreover, setting $x = \xi \neq 0$, we find

that $|f(\xi)| = \|\xi\| \|\xi\|$, i.e., the smallest constant M for which the inequality $|f(x)| \leq M\|x\|$ holds is $\|\xi\|$, so that $\|f\| = \|\xi\|$. The case $\xi = 0$ is trivial.

It follows from the reasoning just given that a one-to-one correspondence has been established between functionals f and vectors $\xi \in \mathbf{R}^n$ that preserves the vector-space operations, i.e., is an isomorphism, and also preserves the norms of elements, i.e., is an isometry. Thus the space \mathbf{R}^n is reflexive, since it coincides with its dual: $\mathbf{R}^n = (\mathbf{R}^n)^*$.

2.5.4 The Space l^1 .

Consider the space l^1 of numerical (complex-valued) sequences $x = (x_1, x_2, \dots)$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$. We shall show that its dual space is the space m of bounded numerical sequences: $\xi \in m$, $\xi = (\xi_1, \xi_2, \dots)$, $\sup_{1 \leq k < \infty} |\xi_k| < \infty$. We shall find the general form of a continuous linear functional on l^1 . Just as in the preceding example we conclude that the expression $f(x) = \sum_{i=1}^{\infty} x_i \bar{\xi}_i$ (x_i and ξ_i are the coordinates of the vectors x and ξ respectively) is obviously a continuous linear functional on l^1 as x ranges over the space l^1 . (This series converges absolutely for all $x \in l^1$.)

The continuity of $f(x)$ follows from the inequalities

$$|f(x)| \leq \sup_{1 \leq k < \infty} |\xi_k| \cdot \sum_{i=1}^{\infty} |x_i| = \|\xi\|_m \cdot \|x\|_{l^1},$$

i.e., $\|f\| \leq \|\xi\|_m$.

We shall show conversely that every continuous linear functional f on l^1 has the form $f(x) = \sum_{i=1}^{\infty} x_i \bar{\xi}_i$, where $x = (x_1, x_2, \dots) \in l^1$ and $\xi = (\xi_1, \xi_2, \dots) \in m$. Let $e_1 = (1, 0, \dots)$, $e_2 = (0, 1, 0, \dots)$, ... be the basis of l^1 , i.e., every $x \in l^1$ admits a unique representation in the form $x = \sum_{n=1}^{\infty} x_n e_n$. (The set $\{e_n\}$ is indeed a basis: if $x^k = \sum_{n=1}^k x_n e_n$, then $\|x^k - x\|_{l^1} = \sum_{n=k+1}^{\infty} |x_n| \rightarrow 0$ as $k \rightarrow \infty$. The uniqueness of such an expansion is obvious. Consequently, any vector $x \in l^1$ can be written uniquely in the

*Here $\bar{\xi}_i$ is the complex conjugate of the number ξ_i .

form $x = \sum_{i=1}^{\infty} x_i e_i$.) Let f be a continuous linear functional on l^1 . Then

$$\begin{aligned} f(x) &= f\left(\sum_{i=1}^{\infty} x_i e_i\right) = f\left(\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i e_i\right) = \lim_{n \rightarrow \infty} f\left(\sum_{i=1}^n x_i e_i\right) \\ &= \sum_{i=1}^{\infty} x_i f(e_i) = \sum_{i=1}^{\infty} x_i \bar{\xi}_i. \end{aligned}$$

The numbers $\bar{\xi}_i = f(e_i)$ form a bounded sequence: $|\bar{\xi}_i| = |f(e_i)| \leq \|f\|$. Consequently $\|\xi\|_m = \sup_{1 \leq k < \infty} |\xi_k| \leq \|f\|$, $\xi = (\xi_1, \xi_2, \dots)$. The vector ξ is

uniquely determined by the functional f . Indeed, let $f(x) = \sum_{i=1}^{\infty} x_i \bar{\xi}_i$ and $x = e_i$. Then $f(e_i) = \bar{\xi}_i$, i.e., we find that $\bar{\xi}_i$ is necessarily $f(e_i)$.

From the inequalities obtained above we have also

$$\|f\| = \sup_{1 \leq k < \infty} |\xi_k| = \|\xi\|_m.$$

Thus there exists an isomorphic and isometric correspondence between the continuous linear functionals on l^1 and the elements of the space m , so that $(l^1)^* = m$.

2.5.5. The Space l^p , $p > 1$.

We shall find the general form of a linear functional on the space l^p and the space dual to l^p . Let $f(x)$ be a linear functional on l^p . Since the elements $e_k = \{e_i^k\}$, where $e_i^k = 0$ for $i \neq k$ and $e_k^k = 1$, form a basis in l^p , the expression

$$x = \sum_{i=1}^{\infty} x_i e_i$$

holds for any element $x \in l^p$.

By the linearity of the functional f we have

$$f(x) = \sum_{i=1}^{\infty} x_i f(e_i).$$

We set $f(e_i) = \bar{\xi}_i$ and study the properties of these numbers. Let $\eta_n = \{x_k^{(n)}\}$, where

$$x_k^{(n)} = \begin{cases} |\xi_k|^{q-1} e^{i \arg \xi_k} & \text{for } k \leq n, \\ 0 & \text{for } k > n. \end{cases}$$

(The number q is chosen by the relation $\frac{1}{p} + \frac{1}{q} = 1$.) We obtain

$$f(\eta_n) = \sum_{i=1}^n |\xi_i|^q.$$

Since

$$|f(\eta_n)| \leq \|f\| \|\eta_n\|_{l^p} = \|f\| \left(\sum_{i=1}^n |\xi_i|^{(q-1)p} \right)^{1/p} = \|f\| \left(\sum_{i=1}^n |\xi_i|^q \right)^{1/p},$$

we have

$$\sum_{i=1}^n |\xi_i|^q \leq \|f\| \left(\sum_{i=1}^n |\xi_i|^q \right)^{1/p},$$

from which the inequality

$$\left(\sum_{i=1}^n |\xi_i|^q \right)^{1/q} \leq \|f\|$$

follows for any n . Therefore

$$\left(\sum_{i=1}^{\infty} |\xi_i|^q \right)^{1/q} \leq \|f\|,$$

i.e., $\{\xi_i\} \in l^q$ and the inequality $\|\xi\|_{l^q} \leq \|f\|$ holds.

On the other hand, take an arbitrary sequence $d = \{d_i\} \in l^q$. Then

$$\varphi(x) = \sum_{i=1}^{\infty} d_i x_i, \quad x = \{x_i\} \in l^p$$

is a linear functional on the space l^p . Indeed the linearity of the functional is obvious and its boundedness follows from the relation established using Hölder's inequality:

$$|\varphi(x)| \leq \left(\sum_{i=1}^{\infty} |d_i|^q \right)^{1/q} \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} = \|d\|_{l^q} \|x\|_{l^p}.$$

Thus the formula

$$f(x) = \sum_{i=1}^{\infty} x_i f(e_i) = \sum_{i=1}^{\infty} x_i \bar{\xi}_i, \quad \bar{\xi}_i = f(e_i),$$

gives the general form of a linear functional in the space l^p . We now determine the norm of the functional f . We have

$$|f(x)| = \left| \sum_{i=1}^{\infty} x_i \bar{\xi}_i \right| \leq \left(\sum_{i=1}^{\infty} |\xi_i|^q \right)^{1/q} \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} = \|\xi\|_{l^q} \|x\|_{l^p},$$

i.e., $\|f\| \leq \|\xi\|_{l^q}$. Comparing this inequality with the inequality $\|\xi\|_{l^q} \leq \|f\|$ obtained earlier, we find that $\|f\| = \|\xi\|_{l^q} = \left(\sum_{i=1}^{\infty} |\xi_i|^q \right)^{1/q}$. From this we find that the dual space to l^p is the space l^q , where $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \geq 1$. Consequently we have the equalities

$$(l^p)^* = l^q, \quad (l^p)^{**} = (l^q)^* = l^p,$$

i.e., the space l^p is reflexive.

In particular if we consider the space l^2 , the general form of a continuous linear functional defined on l^2 will be

$$f(x) = \sum_{i=1}^{\infty} x_i \bar{\xi}_i,$$

where $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty$ and $\|f\| = \left(\sum_{i=1}^{\infty} |\xi_i|^2 \right)^{1/2} = \|\xi\|_{l^2}$, $\xi = \{\xi_i\}$. Consequently $(l^2)^* = l^2$.

2.5.6. Weak Convergence in l^p .

As it happens a necessary and sufficient condition for a sequence $\{x_n\}$ of elements $x_n = \{\xi_i^{(n)}\}$ of the space l^p to converge to an element $x_0 = \{\xi_i^{(0)}\} \in l^p$ is that the sequence $\{\|x_n\|\}$ be bounded and that $\xi_i^{(n)} \rightarrow \xi_i^{(0)}$ as $n \rightarrow \infty$ for each index i . In other words weak convergence in l^p means coordinatewise convergence plus boundedness in norm.

We can verify this by remarking that the linear combinations of elements $f_i = \{0, 0, \dots, 0, 1, 0, \dots\}$, $i = 1, 2, \dots$, are everywhere dense in $l^q = (l^p)^*$. Therefore by Proposition 3'' a necessary and sufficient condition for weak convergence of $\{x_n\}$ to x_0 is that the norms be bounded and that $f_i(x_n) = \xi_i^{(n)} \rightarrow f_i(x_0) = \xi_i^{(0)}$ for any i , which was to be proved.

2.6 Compact Sets. Weak-Star Compactness.

We recall that in Sec. 1.2.5 we gave the definition of a compact space (resp. set) as a space (resp. set) every open cover of which has a finite sub-cover. A set was called precompact if its closure is compact.

In this section we continue our study of the properties of compact sets in normed vector spaces.

We begin by proving the following proposition.

THEOREM 10. *In a finite-dimensional normed space \mathbf{X}^n precompactness is equivalent to boundedness.*

PROOF: If the set M is precompact, then \overline{M} is compact and consequently M is bounded. In fact if a compact set were not bounded, there would be a sequence $\{x_n\}$ of points in it such that $\|x_n\| \rightarrow \infty$. Taking a subsequence $\{x_{n_k}\}$ such that $\|x_{n_{k+1}}\| > \|x_{n_k}\| + 1$, we would arrive at a contradiction to the compactness of \overline{M} (since no subsequence of such a subsequence can converge, cf. Theorem 6 of Sec. 1.3.3).

Conversely, let M be bounded. We shall construct a finite ε -net for M . Let $\xi_1, \xi_2, \dots, \xi_n$ be coordinates in \mathbf{X}^n . Since M is bounded, there exists a number C such that $|x_i| < C$ for $i = 1, 2, \dots, n$, and for all $x = (x_1, x_2, \dots, x_n) \in M$. Let R be the radius of the smallest ball in \mathbf{X}^n containing the unit cube. Choose the number K by the inequality $R/K < \varepsilon$. An ε -net can be chosen from the points of the form $(k_1/K, k_2/K, \dots, k_n/K)$, where k_i are integers contained within the limits $-KC \leq k_i \leq KC$. We remark that the number of elements in the net constructed is $(2KC)^n$, i.e., has order $O(\varepsilon^{-n})$ as $\varepsilon \rightarrow 0$, where n is the dimension of the space \mathbf{X}^n . ■

THEOREM 11. *In an infinite-dimensional normed vector space N the unit ball $O = \{x \in N : \|x\| < 1\}$ is not a precompact set.*

PROOF: Suppose, to the contrary, that the ball O is a precompact set and can be covered by a finite number of balls O_1, O_2, \dots, O_M of radius $r < 1$. Consider an n -dimensional linear manifold \mathbf{X}^n in the space N that contains the centers of these balls. Such a linear manifold exists, at least if $n \geq M$.

Let $\hat{O}, \hat{O}_1, \hat{O}_2, \dots, \hat{O}_M$ be the intersection of the balls O, O_1, O_2, \dots, O_M with \mathbf{X}^n . The set \hat{O} is a ball in \mathbf{X}^n of radius 1 and the sets $\hat{O}_1, \hat{O}_2, \dots, \hat{O}_M$ are balls of radius r in \mathbf{X}^n . Suppose the volume* μ of the ball \hat{O} is 1: $\mu(\hat{O}) = 1$. Then we have $\mu(\hat{O}_i) = r^n$, $i = 1, 2, \dots, M$. Since the ball O is contained in the union of the balls O_i , $i = 1, 2, \dots, M$, the inequality $M \cdot r^n \geq 1$ holds, but since $r < 1$, this inequality cannot hold if n is sufficiently large. We have now reached a contradiction. ■

However the following theorem holds.

*For the definition of the volume (measure) of a set in \mathbf{X}^n see the next chapter.

THEOREM 12. *Let N be a separable normed vector space. Then every ball in the dual space N^* is weak-star compact, i.e., from every sequence of linear functionals $\{f_n\}$ with bounded norms it is possible to choose a subsequence converging weak-star to some linear functional f_0 .*

PROOF: We recall that the concepts of weak-star convergence and pointwise convergence coincide for linear functionals, and so by Theorem 3 of Sec. 2.2.1 the dual space N^* is complete in the sense of weak-star convergence. To prove the theorem it suffices to prove that every bounded sequence $\{f_n\}$ of linear functionals contains a subsequence that is fundamental in the sense of weak-star convergence. We shall do this. For simplicity suppose $\|f_n\| \leq 1$ and $x_1, x_2, \dots, x_n, \dots$ is a countable dense subset of N . Since

$$|f_n(x_1)| \leq \|f_n\| \|x_1\| \leq \|x_1\|,$$

the numerical sequence $\{f_n(x_1)\}$ is bounded. We choose a convergent subsequence of it

$$f_{n_1^1}(x_1), f_{n_2^1}(x_1), \dots$$

Since

$$|f_{n_k^1}(x_2)| \leq \|f_{n_k^1}\| \|x_2\| \leq \|x_2\|,$$

the numerical sequence $\{f_{n_k^1}(x_2)\}$ is bounded. We choose a convergent subsequence of it

$$f_{n_1^2}(x_2), f_{n_2^2}(x_2), \dots$$

This process can be continued, choosing subsequences $\{f_{n_k^3}\}$, etc. Each succeeding subsequence is a part of its predecessor and therefore converges on every element on which its predecessor converges.

We now distinguish the so-called "diagonal" subsequence of functionals

$$f_{n_1^1}, f_{n_2^2}, \dots, f_{n_k^k}, \dots$$

This "diagonal" subsequence converges at each element x_1, x_2, \dots of the countable dense subset of N and the norms of the functionals of the sequence are uniformly bounded. Then by Proposition 3' the sequence $\{f_{n_k^k}\}$ is weak-star convergent. ■

Finally we shall prove a theorem that gives a criterion for precompactness of a set in the space $C(K)$, where K is a compact set, i.e., in the space of continuous functions on a compact metric space K with metric ρ . We recall that the norm of a function $f \in C(K)$ is defined by the formula $\|f\| = \max_{x \in K} |f(x)|$.

The family M of functions $f(x) \in C(K)$ is said to be *uniformly bounded* if there exists a constant C such that $|f(x)| \leq C$ for all $f \in M$.

The family M of functions $f(x) \in C(K)$ is said to be *equicontinuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $f \in M$ and all $x, y \in K$ satisfying $\rho(x, y) < \delta$.

THEOREM 13 (Arzelà-Ascoli). *A necessary and sufficient condition for a family of functions $M \subset C(K)$ to be precompact is that it be uniformly bounded and equicontinuous.*

PROOF: Let M be precompact. Then there exists a finite $\varepsilon/3$ -net f_1, \dots, f_N . Since each of the functions f_i , $i = 1, \dots, N$ is continuous on the compact set K , according to Lemma 9 of Sec. 1.2.5 each function f_i is bounded and, consequently, there exists a constant C_i such that $|f_i(x)| \leq C_i$, $i = 1, \dots, N$. Let $C = \max_{1 \leq i \leq N} C_i + \frac{\varepsilon}{3}$. Then for any function $f \in M$ it follows from the property of an $\varepsilon/3$ -net that there exists an index i such that $|f(x) - f_i(x)| \leq \varepsilon/3$ for all $x \in K$, and therefore

$$|f(x)| \leq |f_i(x)| + \frac{\varepsilon}{3} \leq C_i + \frac{\varepsilon}{3} \leq C, \quad x \in K.$$

Thus the family M is uniformly bounded.

Furthermore each of the functions f_i forming the $\varepsilon/3$ -net is continuous on the compact set K , and hence uniformly continuous on K . Therefore for the given number $\varepsilon/3$ there exist numbers δ_i such that

$$|f_i(x_1) - f_i(x_2)| < \varepsilon/3,$$

if $\rho(x_1, x_2) < \delta_i$.

Let $\delta = \min_i \delta_i$. Then for $\rho(x_1, x_2) < \delta$ and any function $f \in M$, if we choose f_i so that $|f(x) - f_i(x)| < \varepsilon/3$, $x \in K$, we have

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f_i(x_1)| + |f_i(x_1) - f_i(x_2)| + |f_i(x_2) - f(x_2)| < \varepsilon.$$

This establishes that the family M is equicontinuous.

Conversely, let M be a uniformly bounded and equicontinuous family of functions.

Choose δ so that the inequality

$$|f(x_1) - f(x_2)| < \varepsilon/3$$

holds for $\rho(x_1, x_2) < \delta$ and all $f \in M$.

Let $\{y_1, y_2, \dots, y_n\} = S$ be a finite δ -net for the compact set K . Consider the set $\tilde{M} = \{f(y_1), f(y_2), \dots, f(y_n)\}$, where f is any function in M . The set \tilde{M} can be regarded as belonging to the space $m = W$ of bounded sequences (cf. Example 5 of Sec. 1.2.1). We recall that the norm of an element $\xi = \{\xi_1, \xi_2, \dots\}$ in such a space is defined by the rule $\|\xi\|_W = \sup_k |\xi_k|$.

In fact we can restrict ourselves to the finite-dimensional linear manifold W^n of sequences $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ with the same norm as in the space m . The set \tilde{M} in W^n is bounded (by the uniform boundedness of the family of functions M) and therefore precompact. Let $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_N$ be a finite $\varepsilon/3$ -net for the set \tilde{M} in the normed space W^n . We remark that each element \tilde{f}_i of this net has the form

$$\tilde{f}_i = \{f_i(y_1), f_i(y_2), \dots, f_i(y_n)\}, \quad i = 1, 2, \dots, N.$$

We shall show that the set of functions f_1, f_2, \dots, f_N is an ε -net for the set M in the space $C(K)$. We begin by noting that the "trace" in W^n of any function f , i.e., $\tilde{f} = \{f(y_1), \dots, f(y_n)\}$, lies at a distance at most $\varepsilon/3$ in the metric of W^n from some element \tilde{f}_i of the $\varepsilon/3$ -net for \tilde{M} . Let us estimate the distance between f and f_i in the metric of $C(K)$. Let x be any element of K and $y_k \in S$ the element of the δ -net S nearest to it. Then $\rho(x, y_k) < \delta$. Consequently

$$|f(x) - f(y_k)| < \varepsilon/3, \quad |f_i(x) - f_i(y_k)| < \varepsilon/3$$

by the choice of δ . In addition

$$|f(y_k) - f_i(y_k)| < \varepsilon/3,$$

since

$$\sup_{1 \leq k \leq n} |f(y_k) - f_i(y_k)| = \|\tilde{f} - \tilde{f}_i\|_{W^n} < \frac{\varepsilon}{3}.$$

Thus

$$|f(x) - f_i(x)| \leq |f(x) - f(y_k)| + |f(y_k) - f_i(y_k)| + |f_i(y_k) - f_i(x)| < \varepsilon,$$

and therefore $\|f - f_i\|_{C(K)} < \varepsilon$.

Consequently the family M has a finite ε -net and so is precompact. ■

EXERCISES

1. In the space l^p , ($p > 1$) the linear functional

$$f(x) = f(\{x_1, x_2, 0, \dots\}) = x_1 + 2x_2$$

is defined on vectors x of the form $\{x_1, x_2, 0, \dots\}$. Find an extension of it to all of l^p whose norm is $(1 + 2^q)^{1/q}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

2. Denote by X the linear manifold of polynomials in the space $C[0, 1]$. Let linear functionals of the form $f(x) = [x(0) + x(1)]/2$ and

$$f_1(x) = \int_0^1 x(t) dt, \quad x \in X$$

be defined for $x \in X$. Do there exist extensions of these functionals to all of $C[0, 1]$? Are these extensions unique?

3. Prove that in the space l^1 the sequence $e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, \dots)$, \dots has neither a strong nor a weak limit.

4. Prove that the unit sphere in l^p , ($p > 1$) is strongly closed. Find the closure of the unit sphere $S = \{x : \|x\| = 1\}$ in l^p in the sense of weak convergence.

5. Verify that the sequence $\{x_n(t) = t^n\}$ has no weak limit in $C[0, 1]$, hence a fortiori no strong limit.

6. Prove that the the space $C[0, 1]$ is nonreflexive.

7. Consider a Banach space B . Prove that if B^* is separable, then B is also separable.

8. Let the linear transformation A map a normed space N_1 onto a normed space N_2 . Prove that if the relations $\|Ax\| \geq m\|x\|$, $m > 0$, hold for all $x \in N_1$, then A^{-1} exists and is bounded.

9. An operator A is defined on the space l^p ($p \geq 1$) by the rule $Ax = A(x_1, x_2, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \dots)$, and $\sup_n |\lambda_n| < \infty$. Does this transformation have an inverse?

10. Prove that if V is an infinite-dimensional normed vector space, then there exists a discontinuous linear functional on it.

11. Prove that a Banach space is reflexive if and only if the ball $\|x\| \leq 1$ is compact in the weak topology.

12. Suppose a linear manifold M is given in a normed vector space N together with an element $x_0 \notin M$ such that $d = \inf_{x \in M} \|x_0 - x\| > 0$. Construct a linear functional f defined everywhere on N such that $f(x) = 0$ for $x \in M$, $f(x_0) = 1$, and $\|f\| = d^{-1}$.

