

## Chapter 4

# The Geometry of Hilbert Space

# The Spectral Theory of Operators

The present chapter is devoted to the spectral theory of operators in Hilbert and Banach spaces. The most important problems of this theory are propositions on the so-called diagonalization of the operators being studied—spectral theorems, propositions on completeness and the basis property of the eigenvectors of operators, properties of the spectrum and eigenvalues. The most intensively studied class of operators is the completely continuous operators and subclasses of them—nuclear operators and Hilbert-Schmidt operators.

The solution of many important problems of the spectral theory of operators is connected with the theory of analytic functions. The reason is that the basic objects that characterize the spectral problem for an operator, such as the resolvent, the characteristic determinant whose zeros are the eigenvalues of the operator, etc., are analytic functions of the spectral parameter in certain regions.

### 1. HILBERT SPACES

In Chapters 1 and 2 we studied topological spaces, metric spaces, topological vector spaces, normed spaces, and certain properties of transformations (mappings) on them. The most refined of the properties presented were obtained in normed and Banach spaces. This is to be expected, since these spaces are the most special cases of the spaces studied.

Still more interesting theorems will be obtained in this chapter in the study of a special case of a Banach space—Hilbert spaces.

Historically mathematicians came to the study of Hilbert spaces through different considerations: the necessity of such spaces was dictated by physics,

the study of integral equations, and certain generalizations of properties of finite-dimensional spaces.

### 1.1. The Geometry of a Hilbert Space

We give the basic definition at the outset.

**DEFINITION 1.** A Hilbert space is a set  $H$  of elements  $f, g, h, \dots$ , possessing the following properties:

1)  $H$  is a vector space, i.e., the operations of addition and multiplication by real or complex numbers are defined (correspondingly  $H$  is called either a real or a complex Hilbert space);

2) an inner product exists in  $H$ , i.e., a numerical function  $(f, g)$  of pairs of arguments  $f$  and  $g$  satisfying the axioms

- a)  $(af, g) = a(f, g)$  for any number  $a$ ;
- b)  $(f + g, h) = (f, h) + (g, h)$ ;
- c)  $(f, g) = \overline{(g, f)}$ , where the line denotes complex conjugation;
- d)  $(f, f) > 0$  if  $f \neq 0$ ;  $(f, f) = 0$  if  $f = 0$ ;

3)  $H$  is a complete metric space with respect to the distance  $\rho(f, g) = \|f - g\|$ ,\* where the norm of an element  $h$  is defined as  $\|h\| = (h, h)^{1/2}$ .

We give several examples.

1. The simplest example of a Hilbert space is the finite-dimensional vector space  $\mathbf{R}^n$  when an inner product is introduced satisfying Axiom 2) of Definition 1.

2. Consider the space  $l^2$ , whose elements are sequences of numbers  $\{\xi_n\}$  such that  $\sum_{n=1}^{\infty} |\xi_n|^2 < \infty$ . The vector-space operations are defined naturally in this space: if  $\xi$  and  $\eta$  belong to  $l^2$  and  $\xi = \{\xi_n\}$ ,  $\eta = \{\eta_n\}$ , then  $\alpha\xi + \beta\eta = \{\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2, \dots\}$ , where  $\alpha$  and  $\beta$  are numbers.

We introduce an inner product of the elements  $\xi$  and  $\eta$  of this space by the formula

$$(\xi, \eta) = \sum_{n=1}^{\infty} \xi_n \cdot \bar{\eta}_n.$$

By the Cauchy-Bunyakovskii inequality (cf. Sec. 1.2.1) we have

$$|(\xi, \eta)| = \left| \sum_{n=1}^{\infty} \xi_n \cdot \bar{\eta}_n \right| \leq \left\{ \sum_{n=1}^{\infty} |\xi_n|^2 \right\}^{1/2} \cdot \left\{ \sum_{n=1}^{\infty} |\eta_n|^2 \right\}^{1/2} < \infty.$$

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\*It will be proved below that the Cauchy-Bunyakovskii inequality holds, from which it will follow that the function  $\rho(f, g) = \|f - g\|$  defines a metric on  $H$ .



Therefore the inner product can be introduced according to this rule. It is also not difficult to verify that all the axioms of an inner product hold. According to Exercise 3 of Sec. 1.3 the space  $l^2$  is complete.

Thus  $l^2$  is a Hilbert space.

3. Consider the set  $L^2[a, b]$  of Lebesgue square-integrable functions on the closed interval  $[a, b]$ . Since sets of measure zero can be neglected in Lebesgue integration, the elements of  $L^2[a, b]$  are actually equivalence classes of functions (two functions being equivalent if they coincide almost everywhere).

In this space we can introduce as linear operations the usual addition of functions and multiplication of a function by a number.

The inner product in this space is introduced according to the rule

$$(f, g) = \int_{[a, b]} f \cdot \bar{g} \, dx, \quad \text{where } f(x), g(x) \in L^2[a, b].$$

It follows from the inequality

$$|f(x) \cdot g(x)| \leq \frac{1}{2} \{ |f(x)|^2 + |g(x)|^2 \}$$

that if  $f$  and  $g$  belong to  $L^2[a, b]$ , then their product  $f \cdot g$  belongs to  $L^1[a, b]$ . The axioms for an inner product can be verified immediately from the properties of the Lebesgue integral. Thus  $L^2[a, b]$  is a Hilbert space.

4. It follows immediately from axioms a) and c) for an inner product that the number  $a$  can be taken out of the inner product sign if it is the coefficient of the first argument and that the number  $a$  can be taken out and conjugated if it is a coefficient of the second argument, i.e.,

$$(af, g) = a(f, g), \quad (f, ag) = \overline{(ag, f)} = \bar{a}(f, g).$$

5. Let  $N$  be a normed space. The question arises: what additional conditions must the norm satisfy in order for it to be given by an inner product? It turns out that a necessary and sufficient condition for this is that the equality

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2)$$

hold for any two elements  $f$  and  $g$ . In one direction this condition is obvious and can be verified immediately. To prove the converse it is necessary to study the function  $(f, g)$  given by the rule

$$(f, g) = \frac{1}{4} (\|f + g\|^2 - \|f - g\|^2)$$

and show that if the original equality holds, then the function  $(f, g)$  satisfies all the axioms of an inner product.

The Cauchy-Bunyakovskii inequality, whose proof will now be given, is of great importance in a Hilbert space  $H$ . Let  $f, g \in H$  and let  $\lambda$  be a real number. Then, if we set  $h = f_\lambda(f, g)g$ , since  $(h, h) \geq 0$ , we have

$$\begin{aligned} 0 \leq (h, h) &= (f + \lambda(f, g)g, f + \lambda(f, g)g) \\ &= (f, f) + 2\lambda|(f, g)|^2 + \lambda^2|(f, g)|^2(g, g). \end{aligned}$$

Consequently this quadratic trinomial (with respect to  $\lambda$ ) cannot have two distinct real roots. Therefore its discriminant is nonpositive, i.e.

$$|(f, g)|^4 - (f, f)|(f, g)|^2(g, g) \leq 0.$$

Thus (even in the case when  $(f, g) = 0$ ) we have

$$|(f, g)|^2 \leq (f, f)(g, g),$$

or

$$|(f, g)| \leq \|f\| \cdot \|g\|.$$

The inequality just obtained is called the *Cauchy-Bunyakovskii inequality*. Except for the trivial case when  $f = 0$  or  $g = 0$ , equality holds in this inequality if and only if  $f = -\lambda(f, g)g$  for some value of  $\lambda$ , i.e., when the vectors  $f$  and  $g$  are collinear.

Using the Cauchy-Bunyakovskii inequality, it is easy to verify that  $\|f\| = (f, f)^{1/2}$  and the distance  $\rho(f, g) = \|f - g\|$  satisfy the axioms of a norm and a distance respectively.

In order to do this it actually suffices to verify only the triangle inequality  $\|f + g\| \leq \|f\| + \|g\|$  and  $\rho(f, g) \leq \rho(f, h) + \rho(g, h)$  (or, what is the same,  $\|f - g\| \leq \|f - h\| + \|g - h\|$ ). But these inequalities are consequences of the Cauchy-Bunyakovskii inequality.

The presence of a metric in a Hilbert space makes it possible to consider the concepts connected with passage to the limit.

We note that the inner product  $(f, g)$  is a jointly continuous function of the variables  $f$  and  $g$ . Indeed by the Cauchy-Bunyakovskii inequality we have

$$\begin{aligned} |(f, g) - (f_n, g_n)| &= |(f, g) - (f - k_n, g - h_n)| \\ &= |(f, h_n) + (k_n, g) - (k_n, h_n)| \\ &\leq \|f\| \cdot \|h_n\| + \|g\| \cdot \|k_n\| + \|k_n\| \cdot \|h_n\| \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$



where

$$g - g_n = h_n, \quad f - f_n = k_n, \quad f_n \rightarrow f, \quad g_n \rightarrow g, \quad n \rightarrow \infty.$$

Therefore  $(f_n, g_n) \rightarrow (f, g)$  as  $f_n \rightarrow f$  and  $g_n \rightarrow g$ , which was to be proved.

The inner product makes it possible to introduce into  $H$  the concept of the *cosine of the angle* between two nonzero vectors (when  $H$  is a real space):

$$\cos(\widehat{f, g}) = \frac{(f, g)}{\|f\| \cdot \|g\|}.$$

The concept of the cosine of an angle in turn makes it possible to call two vectors *orthogonal* if the cosine of the angle between them is zero.

In other words, the vectors  $f$  and  $g$  of a real or complex space are called orthogonal if  $(f, g) = 0$ . To denote the orthogonality of two vectors we use the symbol  $f \perp g$ .

If  $M$  and  $N$  are subsets (or subspaces) of  $H$ , the symbol  $M \perp N$  means that  $f \perp g$  for any  $f \in M$  and  $g \in N$ .

We remark that if the vector  $f$  is orthogonal to the vectors  $g_1, \dots, g_n$ , then it is orthogonal also to a linear combination of them  $\sum_{i=1}^n \alpha_i g_i$ . If the vectors  $g_1, \dots, g_n, \dots$  are orthogonal to the vector  $f$  and  $g = \lim_{n \rightarrow \infty} g_n$ , then the vector  $g$  is also orthogonal to the vector  $f$ .

Indeed, by the continuity of the inner product we have

$$(g, f) = \lim_{n \rightarrow \infty} (g_n, f) = 0,$$

which was to be proved.

It follows from what has been said that the set of vectors orthogonal to the vectors  $\{f_\alpha\}$  forms a closed linear manifold\* called the *orthogonal complement* of the set  $\{f_\alpha\}$ .

If the vectors  $f$  and  $g$  are orthogonal, then the equality  $\|f + g\|^2 = (f + g, f + g) = \|f\|^2 + \|g\|^2$ —the Pythagorean theorem—is easily verified, or even the more general relation

$$\|f\|^2 = \|f_1\|^2 + \dots + \|f_n\|^2, \quad \text{if } f = f_1 + f_2 + \dots + f_n, \quad f_i \perp f_j, \quad i \neq j.$$

Orthogonal systems of vectors play an important role in Hilbert space. To obtain such a system an orthogonalization method known as the Gram-Schmidt process is applied.

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\*We emphasize that the entire space is a closed linear manifold.

We take a sequence of elements  $\{f_n\}$ ,  $f_n \neq 0$ , of  $H$ , each of which is linearly independent of its predecessors.

We shall show that it can be replaced by an orthonormal sequence  $\{\varphi_n\}$ , i.e., a sequence such that  $(\varphi_n, \varphi_n) = \delta_{nk}$ , each element  $\varphi_n$  is a linear combination of the elements  $f_m$  with indices  $m \leq n$ , and conversely each  $f_n$  is a linear combination of the elements  $\varphi_m$  with  $m \leq n$ :

$$\begin{aligned}\varphi_n &= c_{n1} f_1 + c_{n2} f_2 + \cdots + c_{nn} f_n, \\ f_n &= \gamma_{n1} \varphi_1 + \gamma_{n2} \varphi_2 + \cdots + \gamma_{nn} \varphi_n.\end{aligned}$$

We carry out the proof by induction. We set  $\varphi_1 = f_1 / \|f_1\|$ . Then  $c_{11}^{-1} = \gamma_{11} = \|f_1\|$ . We then subtract from  $f_2$  a scalar multiple of  $\varphi_1$  chosen so that the difference  $h_2 = f_2 - \gamma \varphi_1$  is orthogonal to the vector  $\varphi_1$ :  $(h_2, \varphi_1) = (f_2, \varphi_1) - \gamma(\varphi_1, \varphi_1) = 0$ . Consequently  $\gamma$  must be chosen to be  $(f_2, \varphi_1)$ .

Since  $f_2$  and  $f_1$ , and therefore also  $f_2$  and  $\varphi_1$ , are linearly independent, it follows that  $h_2 \neq 0$  and we set  $\varphi_2 = h_2 / \|h_2\|$ . When this is done, as is easy to see,  $\varphi_2$  turns out to be a linear combination of  $f_2$  and  $f_1$  and conversely  $f_2$  can be expressed linearly in terms of  $\varphi_2$  and  $\varphi_1$ . The construction of the system  $\{\varphi_n\}$  mentioned above is easy to complete by induction.

We give the following definitions:

**DEFINITION 2.** We call a system in a Hilbert (or Banach) space *complete* in  $H$  if it generates the whole space, i.e., if an arbitrary element of  $H$  can be approximated with arbitrary precision in the norm by linear combinations of elements of the system.

**DEFINITION 3.** A Hilbert space  $H$  is called *separable* if it contains a countable everywhere dense set, i.e., a set whose closure in the metric of  $H$  coincides with the whole space  $H$ .

We remark that if a space is separable, then there naturally exists in it a countable complete system (the elements of a countable dense set form a countable complete system). The converse is also true: if there exists a countable complete system in  $H$ , then the space  $H$  is separable. Indeed, in this case any vector can be approximated by a linear combination of the complete system, and the coefficients of the approximating vector can be approximated by numbers with rational components, i.e., numbers of the form  $\gamma = \alpha + i\beta$ , where  $\alpha$  and  $\beta$  are rational numbers. The set of such linear combinations is countable and everywhere dense in  $H$ . Consequently  $H$  is separable.

It is easy to see that if a given system  $\{\varphi_n\}$  is complete, then there is no nonzero vector in  $H$  orthogonal to all the vectors of the system. Indeed, suppose that for some vector  $g \neq 0$  the equality  $(g, \varphi_k) = 0$  holds for  $k =$



1, 2, .... The orthogonal complement of the vector  $g$  then contains all the vectors  $\varphi_k$ , all linear combinations of them, and the closure of the set of these linear combinations, i.e., the entire space  $H$ . In particular  $(g, g) = 0$ , whence  $g = 0$ , contrary to hypothesis.

If the system  $\{\varphi_n\}$  is obtained by orthogonalizing some system  $\{f_n\}$ , then by the orthogonalization formulas it is easy to see that the completeness of the system  $\{\varphi_n\}$  can be proved by proving that the linear combinations of the vectors of the original system are dense in  $H$ .

## 1.2. Bases of a Hilbert Space

In studying complete systems  $\{e_i\}$  it is a very important question whether a given system forms a basis of a separable space, i.e., whether any element  $x$  of the space can be represented in the form

$$x = \sum_{i=1}^{\infty} \xi_i e_i$$

and in only one way (here  $\xi_i$  are numbers and the series converges in the norm of the space).

We remark that the definition of a basis given above carries over precisely for separable Banach spaces. It is of interest to note that, although bases have been constructed for all the basic separable Banach spaces, the question whether there exists a basis in an arbitrary separable Banach space happens to be complicated, and a negative answer has been given only recently.\*

We note that if the system  $\{e_i\} \subset E$ , where  $E$  is a Banach space, is complete and contains no linearly dependent elements, it still does not follow that it is a basis. Indeed, take for example the space  $C[0, 1]$ . The sequence  $\{t^k\}$   $k = 0, 1, \dots$ , is complete in this space by the Weierstrass approximation theorem, but is not a basis. In fact if a function  $f(t) = \sum_{k=0}^{\infty} c_k t^k$  is representable by a uniformly convergent series, then  $f(t)$  is analytic for  $|t| < 1$ . It is clear that such functions do not exhaust the whole space  $C[0, 1]$ , and so the system  $\{t^k\}$  is not a basis in  $C[0, 1]$ .

The sequence  $\{t^k\}$  is also not a basis in the Hilbert space  $L^2[0, 1]$  (the Lebesgue square-integrable functions with  $(f, g) = \int_{[0, 1]} f \bar{g} dx$ ). Let  $f \in$

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\*In 1972 M. Enflo constructed a reflexive separable Banach space having no basis.

$L^2[0, 1]$  and  $f = \sum_{k=0}^{\infty} c_k t^k$ , where the series converges in the metric of  $L^2[0, 1]$ .

We multiply both sides of this equality by the function

$$g(t) = \begin{cases} 1, & t \leq s, \\ 0, & t > s, \end{cases}$$

and integrate. For any  $s$  with  $0 \leq s \leq 1$  we obtain the equality

$$\int_0^s f(t) dt = \sum_{k=0}^{\infty} \frac{c_k}{k+1} s^{k+1},$$

from which it follows that the function  $F(s) = \int_0^s f(t) dt$  is analytic for  $|s| < 1$  and so the function  $f(t)$  is also analytic for  $|t| < 1$ .

For a separable Hilbert space, however, it turns out that a complete orthonormal system is a basis. Moreover for the coefficients  $\xi_i$ , which are uniquely determined, the equality  $\|x\| = \sum |\xi_i|^2$  holds. In this case the coefficients  $\xi_i$  are easy to determine, since the system  $\{e_i\}$  is orthonormal and so the relation  $x = \sum_{i=1}^{\infty} \xi_i e_i$  implies that

$$(x, e_k) = \left( \lim_{n \rightarrow \infty} \sum_{i=1}^n \xi_i e_i, e_k \right) = \lim_{n \rightarrow \infty} \left( \sum_{i=1}^n \xi_i e_i, e_k \right) = \xi_k.$$

The following theorem holds.

**THEOREM 1.** *In a separable Hilbert space  $H$  every complete orthonormal system  $\{\varphi_n\}$  is a basis, i.e., for any  $f \in H$  the expansion*

$$f = \sum_{n=1}^{\infty} (f, \varphi_n) \varphi_n$$

*holds, and  $\|f\|^2 = \sum_{n=1}^{\infty} |(f, \varphi_n)|^2$  (Parseval's identity).*

**PROOF:** We shall prove that the series  $\sum_{n=1}^{\infty} |(f, \varphi_n)|^2$  converges. We form the vector  $g = \sum_{n=1}^p (f, \varphi_n) \varphi_n$ , and let  $f = g + h$ , where  $h$  is to be determined. The vector  $h$  is orthogonal to each of the vectors  $\varphi_1, \dots, \varphi_p$  and consequently also to their linear span. Indeed

$$\begin{aligned} (h, \varphi_j) &= (f, \varphi_j) - (g, \varphi_j) = (f, \varphi_j) - \left( \sum_{n=1}^p (f, \varphi_n) \varphi_n, \varphi_j \right) \\ &= (f, \varphi_j) - (f, \varphi_j) = 0, \quad j = 1, 2, \dots, p. \end{aligned}$$



By the Pythagorean theorem

$$\|f\|^2 = \|g\|^2 + \|h\|^2 = \sum_{n=1}^p |(f, \varphi_n)|^2 + \|h\|^2 \geq \sum_{n=1}^p |(f, \varphi_n)|^2.$$

Thus  $\sum_{n=1}^p |(f, \varphi_n)|^2 \leq \|f\|^2$  for any  $p$ . Passing to the limit as  $p \rightarrow \infty$ , we obtain the so-called *Bessel's inequality*:

$$\sum_{n=1}^{\infty} |(f, \varphi_n)|^2 \leq \|f\|^2.$$

We now introduce the abbreviation  $(f, \varphi_n) = \xi_n$ . Let  $s_p = \sum_{n=1}^p \xi_n \varphi_n$ . Then

$$\|s_p - s_q\|^2 = \sum_{n=p+1}^{\infty} |\xi_n|^2, \text{ for } q > p. \text{ As } p \rightarrow \infty \text{ this quantity tends to}$$

zero, since the numerical series  $\sum_{n=1}^{\infty} |\xi_n|^2$  converges (cf. Bessel's inequality).

Therefore the sequence  $\{s_p\}$  is fundamental and by the completeness of  $H$  it converges:  $\lim_{p \rightarrow \infty} s_p = s \in H$ . We shall show that  $s = f$ . To do this we remark that for a fixed  $k$  and all  $p > k$  the relation  $(s, \varphi_k) = \lim_{p \rightarrow \infty} (s_p, \varphi_k) = \lim_{p \rightarrow \infty} \left( \sum_{n=1}^p \xi_n \varphi_n, \varphi_k \right) = \xi_k = (f, \varphi_k)$ . Therefore for any  $k$  we have  $(f - s, \varphi_k) = (f, \varphi_k) - (s, \varphi_k) = 0$ ; since the system  $\{\varphi_n\}$  is complete, it follows that  $f = s$ , i.e.

$$f = \lim_{p \rightarrow \infty} s_p = \sum_{n=1}^{\infty} \xi_n \varphi_n.$$

Since the inner product is continuous, we find

$$\begin{aligned} \|f\|^2 &= (f, f) = \left( \lim_{p \rightarrow \infty} s_p, \lim_{p \rightarrow \infty} s_p \right) \\ &= \lim_{p \rightarrow \infty} (s_p, s_p) = \lim_{p \rightarrow \infty} \sum_{n=1}^p |\xi_n|^2 = \sum_{n=1}^{\infty} |\xi_n|^2. \blacksquare \end{aligned}$$

**REMARK 1.** If  $g$  is any other vector of the space, then obviously  $(f, g) = \sum_{n=1}^{\infty} (f, \varphi_n) \cdot \overline{(g, \varphi_n)} = \sum_{n=1}^{\infty} \xi_n \cdot \bar{\eta}_n = (\xi, \eta)_{l^2}$ , where  $\eta_n = (g, \varphi_n)$  and  $g = \sum_{n=1}^{\infty} \eta_n \varphi_n$ . Moreover  $\xi = \{\xi_n\}, \eta = \{\eta_n\} \in l^2$ .

REMARK 2. If  $\xi_1, \dots, \xi_n, \dots$  is any sequence of numbers such that  $\sum_{n=1}^{\infty} |\xi_n|^2 < \infty$ , then the series  $\sum_{n=1}^{\infty} \xi_n \varphi_n$  converges in  $H$ . If we denote the sum by  $f$ , then, taking the inner product of  $f$  with  $\varphi_k$ , we shall obviously have  $\xi_k = (f, \varphi_k)$ . Hence there is a one-to-one correspondence between the set of sequences of numbers the series of whose squared moduli converges, i.e., the space  $l^2$ , and the vectors of the Hilbert space  $H$ . This correspondence obviously preserves the vector space operations. In the vector space consisting of all sequences of numbers  $\{\xi_n\}$  such that  $\sum_{n=1}^{\infty} |\xi_n|^2 < \infty$ , as stated in Example 2 above, one can introduce an inner product by the rule

$$(\xi, \eta) = \sum_{n=1}^{\infty} \xi_n \cdot \bar{\eta}_n.$$

It follows from Remark 1 that the one-to-one correspondence between the vectors of an arbitrary separable Hilbert space  $H$  and the vectors of the Hilbert space  $l^2$  (under which the vector  $f \in H$  corresponds to the vector  $\xi = \{\xi_n\} \in l^2$  whose coordinates are the Fourier coefficients of the vector  $f$  in the orthonormal system  $\{\varphi_n\}$ , i.e.,  $f = \sum_{n=1}^{\infty} \xi_n \varphi_n$ ) preserves the inner product also, i.e.,  $(f, g)_H = (\xi, \eta)$ .

Thus any two separable Hilbert spaces are *isomorphic to* (in a one-to-one correspondence that preserves the vector space operations and the inner product with) the space  $l^2$  and, consequently isomorphic to each other. We shall not distinguish such spaces.

In particular the space  $L^2[a, b]$  whose elements are equivalence classes of Lebesgue square-integrable functions (cf. Example 3) is a Hilbert space isomorphic to the space  $l^2$ . The inner product in this space, as we know, was introduced according to the rule

$$(f, g) = \int_a^b f(x) \overline{g(x)} dx, \quad f(x), g(x) \in L^2[a, b],$$

where  $\overline{g(x)}$  denotes the equivalence class of functions that are complex conjugates of the equivalence class of functions  $g(x)$  (this is a class of functions any two of which coincide almost everywhere).

The space  $L^2[a, b]$ , as we know from the considerations of Sec. 3.3, is a complete space. It is obviously also separable. A countable everywhere-dense set in it can be taken, for example, to be the polynomials with rational coefficients. These polynomials are dense in  $C[a, b]$  and the continuous functions are dense in  $L^2[a, b]$ . Hence the spaces  $L^2[a, b]$  and  $l^2$  are isomorphic and



may be identified, since they are merely different realizations of an abstract separable Hilbert space  $H$ .

We shall now pause to give more details about the properties of complete orthonormal systems in separable Hilbert spaces.

It was noted above that if a given orthonormal system is complete in the Hilbert space  $H$ , there is no nonzero vector in  $H$  orthogonal to all the vectors of the system.

It turns out that the converse assertion is also true. If an orthonormal system  $\{\varphi_n\}$  is given in a separable Hilbert space  $H$  and has the property that the only vector orthogonal to all vectors of the system is the zero vector, then the system is complete in  $H$ .<sup>\*</sup> Indeed, suppose  $\{\varphi_n\}$  is not complete in  $H$ , i.e., Parseval's equality does not hold for it. Since Bessel's inequality holds for any function, the failure of Parseval's equality means that there exists a vector  $g$  such that  $\|g\|^2 > \sum_{k=1}^{\infty} |\xi_k|^2$ ,  $\xi_k = (g, \varphi_k)$ . The series  $\sum_{k=1}^{\infty} |\xi_k|^2$  converges, and so, according to Remark 2 following Theorem 1, there exists a vector  $f \in H$  such that  $(f, \varphi_k) = \xi_k$  and  $\sum_{k=1}^{\infty} (f, \varphi_k) \varphi_k = f$ . But then

$\|f\|^2 = \sum_{k=1}^{\infty} |\xi_k|^2$ . The vector  $f - g$  is orthogonal to all the vectors  $\varphi_k$ , but

$\|f\|^2 = \sum_{k=1}^{\infty} |\xi_k|^2 < \|g\|^2$ . Therefore  $\|f\|^2 - \|g\|^2 < 0$ , and so  $\|f\| \neq \|g\|$ ,

whence  $h = f - g \neq 0$ . Thus there exists a vector  $h \neq 0$  orthogonal to all the vectors  $\varphi_k$ , and we have reached a contradiction to our hypothesis. Thus we have proved the following proposition.

**PROPOSITION.** *A necessary and sufficient condition for an orthonormal system  $\{\varphi_k\}$  in a separable Hilbert space to be complete is that only the zero vector be orthogonal to all the vectors of the system  $\{\varphi_k\}$ .*

### 1.3. The Dimension of a Hilbert Space

On the one hand a separable Hilbert space is a vector space and has an algebraic basis, while on the other hand the presence of the inner product makes it possible to consider an orthogonal basis. For that reason the concept of the dimension of a Hilbert space has two different meanings: dimension defined as the cardinality of a set of elements constituting an algebraic basis (the *algebraic dimension*) and the cardinality of the elements of a complete orthonormal system  $\{\varphi_\alpha\}$ . (It will be shown below that the

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<sup>\*</sup>That is, by the assertion of Theorem 1, the system  $\{\varphi_n\}$  is a basis of the space  $H$ .

latter is independent of the choice of the orthonormal system  $\{\varphi_\alpha\}$ .) This dimension is called the *orthogonal dimension*.

In a separable Hilbert space, as follows from Theorem 1, there is an orthonormal basis (having countable cardinality\*). Indeed, carrying out the Gram-Schmidt orthogonalization procedure on a countable system that is complete in  $H$ , we obtain an orthonormal system that is again countable and complete, i.e., a basis. On the other hand, if there is an orthonormal basis in  $H$ , then there is also a countable everywhere-dense set. Let  $\{\varphi_n\}_{n=1}^\infty$  be an orthonormal basis and  $M$  the set of vectors of the form  $\sum_{k=1}^n \gamma_k^{(n)} \varphi_k$ ,  $\gamma_k^{(n)} = \alpha_k^{(n)} + i\beta_k^{(n)}$ , where  $\alpha_k^{(n)}$  and  $\beta_k^{(n)}$  are rational numbers. For any  $h \in H$  and any  $\varepsilon > 0$  one can find  $n$  such that  $\|h - \sum_{k=1}^n (h, \varphi_k) \varphi_k\| < \frac{\varepsilon}{2}$ , and then replace the numbers  $(h, \varphi_k)$  by numbers  $\gamma_k^{(n)}$  so close to them that  $\|\sum_{k=1}^n \{(h, \varphi_k) - \gamma_k^{(n)}\} \varphi_k\| < \frac{\varepsilon}{2}$ . Thus the vector  $f_n = \sum_{k=1}^n \gamma_k^{(n)} \varphi_k$  belongs to the countable set  $M$  and approximates the vector  $h \in H$  with precision up to  $\varepsilon$ :  $\|h - f_n\| < \varepsilon$ . Therefore the space  $H$  is separable. That is, the separable Hilbert spaces and only the separable ones, have a countable orthonormal basis.

The cardinalities of any two complete orthonormal systems in a separable space are the same. Indeed, the fact that a separable space has a countable orthonormal system was shown above. If we suppose that there is also an uncountable orthonormal system, then a countable system would be insufficient to approximate each of its elements with arbitrary precision. Hence the space would not be separable in that case, which is a contradiction.

Thus any complete orthonormal system in a separable Hilbert space has countable cardinality and the separable Hilbert spaces are called countable-dimensional (the orthogonal dimension being meant). Actually in any, not necessarily separable, Hilbert space all orthonormal systems have the same cardinality, which is called the orthogonal dimension of the space. We shall not give a proof of this fact, but only an example of a nonseparable Hilbert space.

Consider the set of functions of the form  $e^{i\lambda t}$ ,  $-\infty < t < +\infty$ , where the parameter  $\lambda$  belongs to  $\mathbf{R}^1$ . Let  $V$  be the span of this set, i.e., the elements of the form  $f_M = \sum_{k=1}^M a_k e^{i\lambda_k t}$ . We define the scalar product of two

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\*In Sec. 4.1.2 the assumption of countable cardinality was part of the definition of a basis.



such elements by the rule

$$\begin{aligned}(f_M, g_N) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f_M \cdot \bar{g}_N dt = \lim_{T \rightarrow \infty} \sum_{k,p=1}^{M,N} a_k \bar{b}_p \frac{1}{2T} \int_{-T}^T e^{i\lambda(\lambda_k - \mu_p)} dt \\ &= \sum_{k,p=1}^{M,N} \delta(\lambda_k, \mu_p) a_k \bar{b}_p, \quad \text{where } \delta(\lambda, \mu) = \begin{cases} 0, & \lambda \neq \mu, \\ 1, & \lambda = \mu. \end{cases}\end{aligned}$$

We complete  $V$  in the metric generated by this scalar product. The result is a Hilbert space  $H$ . This space is nonseparable, since it contains a continuum of pairwise orthogonal vectors. According to what was said above, the dimension of this space is the cardinality of the set of real numbers, i.e., a continuum.

#### 1.4. Orthogonal Expansions in Hilbert Space

The following important theorem holds in a Hilbert space  $H$ .

**THEOREM 2.** *Let  $H$  be a Hilbert space and  $V$  a subspace of  $H$ . Every vector  $f \in H$  has a decomposition  $f = g + h$ ,  $g \in V$ ,  $h \perp V$  (i.e., the vector  $h$  is orthogonal to every vector in  $V$ ) and  $g$  and  $h$  are uniquely determined.*

**PROOF:** We use the notation  $d = \inf_{g \in V} \|f - g\|$ . Let  $d = 0$ . Then there exists a sequence  $g_n \in V$  such that  $\|f - g_n\| \rightarrow 0$ , from which it follows that  $f$  is a limit point of  $V$  and so, since  $V$  is closed, we have  $f \in V$ . The desired decomposition then has the form  $f = f + 0$ .

Let  $d > 0$ . Consider a sequence  $g_n \in V$  for which  $\|f - g_n\| \rightarrow d$ . By a simple verification we see that the following equality holds:

$$2\|f - g_n\|^2 + 2\|f - g_m\|^2 = 4\left\|f - \frac{g_n + g_m}{2}\right\|^2 + \|g_n - g_m\|^2.$$

As  $n, m \rightarrow \infty$  the left-hand side tends to  $4d^2$ . The first term on the right-hand side is at least  $4d^2$ , since  $\frac{g_n + g_m}{2} \in V$ , so that  $\left\|f - \frac{g_n + g_m}{2}\right\| \geq d$ . Consequently  $\|g_n - g_m\| \rightarrow 0$ . Therefore the sequence  $\{g_n\}$  is fundamental, and since  $H$  is complete, it converges:  $g = \lim_{n \rightarrow \infty} g_n$ , and  $g \in V$ , since  $V$  is closed in  $H$ . Let  $h = f - g$ . Then  $h \perp V$ . Indeed, for any vector  $l \in V$  and any number  $\lambda$

$$\begin{aligned}d^2 &\leq \|f - (g - \lambda l)\|^2 = \|h + \lambda l\|^2 = (h + \lambda l, h + \lambda l) \\ &= d^2 + \bar{\lambda}(h, l) + \lambda(l, h) + |\lambda|^2 \|l\|^2.\end{aligned}$$

From this we find that  $\bar{\lambda}(h, l) + \lambda(l, h) + |\lambda|^2 \|l\|^2 \geq 0$ . But this cannot happen for all  $\lambda$  unless  $(h, l) = (l, h) = 0$ . Indeed it suffices to set  $\lambda = t \cdot e^{i \arg(h, l)}$ , where  $t$  is real.

The decomposition  $f = g + h$  is unique. Suppose  $f = g + h = g' + h'$ , where  $g$  and  $g'$  belong to  $V$  and  $h$  and  $h'$  are orthogonal to  $V$ . We have then  $0 = (g - g') + (h - h')$ , where  $(g - g') \in V$  and  $(h - h') \perp V$ . By the Pythagorean theorem then  $g - g' = h - h' = 0$ , from which  $g = g'$  and  $h = h'$ . ■

The set of vectors  $h$  orthogonal to a subspace  $V$  forms a (closed) subspace  $M$ , which is called the *orthogonal complement* of the subspace  $V$ .

Thus it has been proved that every subspace  $V \subset H$  has an orthogonal complement  $M$  and every vector  $f \in H$  has a decomposition  $f = g + h$ ,  $g \in V$ ,  $h \in M$ .

The space  $H$  is said to be the direct sum of the subspaces  $V$  and  $M$ , and we write  $H = V \oplus M$  or  $V = H \ominus M$ , where the symbols  $\oplus$  and  $\ominus$  mean that the sum and difference are "direct," i.e.,  $f = g + h$ ,  $g \in V$ ,  $h \in M$ ,  $g \perp h$ , or  $g = f - h$ ,  $g \perp h$ , and these representations are unique.

### 1.5. Biorthogonal Sequences

Suppose two sequences  $\{f_k\}_{k=1}^{\infty}$  and  $\{g_k\}_{k=1}^{\infty}$  are given in the Hilbert space  $H$ .

DEFINITION 4. The pair of sequences

$$\{f_k\}, \{g_k\}, \quad k = 1, 2, \dots, \infty,$$

in  $H$  forms a *biorthogonal system* if

$$(f_j, g_k) = \delta_{jk}.$$

Not every linearly independent sequence of vectors has a biorthogonal sequence. For example, in  $L^2[0, 1]$  the sequence  $\{t^k\}_{k=0}^{\infty}$  is linearly independent, but there is no biorthogonal sequence for it. Indeed, if such a sequence  $\{g_k\}_{k=0}^{\infty}$  existed, the following equalities would hold:

$$\int_0^1 g_m(t) t^m dt = 1, \quad \int_0^1 g_m(t) t^k dt = 0, \quad k \neq m.$$

But then we would have  $\int_0^1 \{g_m(t) t^{m+1}\} t^j dt = 0$ ,  $j = 0, 1, \dots$ . However, since the system  $\{t^k\}_{k=0}^{\infty}$  is complete in  $l^2[0, 1]$ ,\* (by Weierstrass' theorem

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\*That is, any element  $f \in L^2[0, 1]$  can be approximated with arbitrary precision in the norm of  $L^2[0, 1]$  by a linear combination of elements of the system.



the system  $\{t^k\}_{k=0}^\infty$  is complete in  $C[0, 1]$ , and the continuous functions are dense in  $L^2[0, 1]$ , we would find that  $g_m(t)t^{m+1} \equiv 0$ , i.e.,  $g_m(t)t^m \equiv 0$ , so that  $\int_0^1 g_m(t)t^m dt = 0$ , contrary to the assumption  $\int_0^1 g_m(t)t^m dt = 1$ .

We now take up a more detailed study of biorthogonal systems.

We shall prove a criterion for the existence of a system biorthogonal to a given system.

**THEOREM 3.** *A sequence of vectors  $\{f_k\}$  has a biorthogonal sequence if and only if for each  $j$  the vector  $f_j$  does not belong to the closure of the linear manifold spanned by the vectors  $f_1, f_2, \dots, f_{j-1}, f_{j+1}, \dots$  (A system satisfying this condition is called minimal.) If the system is minimal and complete, there exists a unique sequence biorthogonal to it.*

**PROOF. SUFFICIENCY:** Let the system  $\{f_k\}$  be minimal. We denote the closure of the linear manifold spanned by  $f_1, \dots, f_{j-1}, f_{j+1}, \dots$  by  $M_j$  and the closure of the linear manifold spanned by all the vectors  $f_k$ ,  $k = 1, 2, \dots$  by  $M$ . By hypothesis, for each natural number  $j$  the relation  $M_j \neq M$  holds. Let  $N_j = M \ominus M_j$  be the orthogonal complement to  $M_j$  in  $M$ . Take a vector  $g_j \in N_j$ . Such a vector  $g_j$  obviously exists. We normalize it by the condition

$$(f_j, g_j) = 1.$$

It is obvious that  $(f_j, g_k) = 0$  for  $j \neq k$ , since  $g_k$  belongs to  $N_k$ , the orthogonal complement to the subspace spanned by  $\{f_1, f_2, \dots, f_{k-1}, f_{k+1}, \dots\}$ . Therefore the sequence  $\{g_k\}$  is biorthogonal to  $\{f_k\}$ .

**NECESSITY:** If the sequence  $\{g_k\}$  is biorthogonal to  $\{f_k\}$ , then for any  $j$  the vector  $f_j$  cannot belong to  $M_j$ . Indeed, if it did, it would be orthogonal to the vector  $g_j$  (since  $g_j$  is orthogonal to  $M_j = \{f_1, f_2, \dots, f_{j-1}, f_{j+1}, \dots\}$  because of the biorthogonality of the systems  $\{g_k\}$  and  $\{f_k\}$ :  $(g_j, f_k) = 0$  for  $k \neq j$ ). Thus we would find that  $(g_j, f_j) = 0$  also, which is impossible by the biorthogonality of the systems  $\{g_k\}$  and  $\{f_k\}$ .

Finally suppose the system  $\{f_k\}$  is minimal and complete. Then  $N_j$  is one-dimensional (otherwise the system would not be complete). The element  $g_j$  in this case is uniquely determined by the conditions  $g_j \in N_j$  and  $(g_j, f_j) = 1$ . ■

The biorthogonal system makes it possible to determine the coefficients of an expansion in a basis. Indeed if  $f = \sum_{k=1}^\infty c_k f_k$  and  $\{g_j\}$  is biorthogonal to  $\{f_k\}$ , then  $(f, g_j) = c_j$ .

We now consider a Banach space and determine the sequence biorthogonal to a basis. Let the sequence  $\{\varphi_n\}$  be a basis of a separable Banach

space  $B$ . Consider the numerical sequences  $y = \{c_1, \dots, c_n, \dots\}$  such that the series  $\sum_{n=1}^{\infty} c_n \varphi_n$  converges in the norm of the space  $B$ . The set of all such sequences obviously forms a vector space  $B_1$ . We introduce a norm into it by setting

$$\|y\| = \sup_n \left\| \sum_{i=1}^n c_i \varphi_i \right\|.$$

The fulfillment of all the axioms of a normed space is verified without difficulty. We shall show that it is a complete (i.e., Banach) space. If  $\{y_m\}$  is a fundamental sequence and

$$B_1 \ni y_m = \{c_1^m, \dots, c_n^m, \dots\},$$

then

$$\|y_m - y_k\|_{B_1} = \sup_n \left\| \sum_{i=1}^n (c_i^m - c_i^k) \varphi_i \right\|_B < \varepsilon \quad \text{for } m, k \geq N(\varepsilon)$$

and therefore

$$\left\| \sum_{i=1}^n (c_i^m - c_i^k) \varphi_i \right\| < \varepsilon, \quad m, k \geq N(\varepsilon), \quad \text{for any } n.$$

Hence

$$\|(c_n^m - c_n^k) \varphi_n\| = \left\| \sum_{i=1}^n (c_i^m - c_i^k) \varphi_i - \sum_{i=1}^{n-1} (c_i^m - c_i^k) \varphi_i \right\| < 2\varepsilon.$$

Therefore

$$|c_n^m - c_n^k| \leq \frac{2\varepsilon}{\|\varphi_n\|}, \quad m, k \geq N(\varepsilon), \quad \text{for any } n.$$

Consequently the numerical sequence  $\{c_n^m\}$ ,  $m = 1, 2, \dots$  converges to some limit  $c_n^*$ , and this holds for any  $n$ . We now pass to the limit in the inequality

$$\left\| \sum_{i=1}^n (c_i^m - c_i^k) \varphi_i \right\| < \varepsilon. \quad \text{We have}$$

$$\lim_{k \rightarrow \infty} \left\| \sum_{i=1}^n (c_i^m - c_i^k) \varphi_i \right\| \leq \varepsilon, \text{ i.e., } \left\| \sum_{i=1}^n (c_i^m - c_i^*) \varphi_i \right\| \leq \varepsilon.$$

We set

$$s_n^m = \sum_{i=1}^n c_i^m \varphi_i, \quad s_n^* = \sum_{i=1}^n c_i^* \varphi_i.$$



Taking account of the preceding inequality, we have

$$\|s_{n+p}^* - s_n^*\| \leq \|s_{n+p}^m - s_n^m\| + 2\varepsilon \quad \text{for } m \geq N(\varepsilon) \text{ and any } n.$$

Suppose some  $\delta > 0$  is given. Choosing  $\varepsilon > 0$  such that  $2\varepsilon < \delta/2$  and fixing  $N(\varepsilon)$ , we choose  $M_0$  so that

$$\|s_{n+p}^m - s_n^m\| < \frac{\delta}{2} \quad \text{for } n \geq M_0 \text{ and any } p > 0$$

(this is possible by the convergence of the series  $\sum_{n=1}^{\infty} c_n^m \varphi_n$ ). Then  $\|s_{n+p}^* - s_n^*\| < \delta$  for  $n \geq M_0$ , i.e., the series  $\sum_{n=1}^{\infty} c_n^* \varphi_n$  converges, and therefore  $y^* = \{c_1^*, \dots, c_n^*, \dots\} \in B_1$ . In addition we have  $\sup_n \left\| \sum_{i=1}^n (c_i^m - c_i^*) \varphi_i \right\| \leq \varepsilon$  for  $m \geq N(\varepsilon)$ , i.e.,  $\|y_m - y^*\|_{B_1} \leq \varepsilon$  for  $m \geq N(\varepsilon)$ , and the completeness of the space  $B_1$  is proved.

It is obvious that to each  $x = \sum_{i=1}^{\infty} \xi_i \varphi_i \in B$  corresponds a unique element  $y_x = \{\xi_1, \dots, \xi_n, \dots\} \in B_1$ ; conversely to each element  $y = \{\xi_i\} \in B_1$  there corresponds a unique element  $x_y \in B$ , namely

$$x_y = \sum_{i=1}^{\infty} \xi_i \varphi_i.$$

Thus we may assume that a transformation  $x = Ay$  is defined mapping  $B_1$  in a one-to-one manner onto  $B$ . It is easy to see that the operator  $A$  is linear. Moreover

$$\|Ay\| = \|x\| = \left\| \sum_{i=1}^{\infty} \xi_i \varphi_i \right\| \leq \sup_n \left\| \sum_{i=1}^n \xi_i \varphi_i \right\| = \|y\|, \quad x = \sum_{i=1}^{\infty} \xi_i \varphi_i,$$

i.e., the operator  $A$  is bounded. By the bounded inverse theorem there exists an inverse operator  $y = A^{-1}x$  that is also bounded and linear.

Let  $x = \sum_{i=1}^{\infty} \xi_i \varphi_i \in B$ . We define a functional  $F_k$  by setting  $F_k(x) = \xi_k$ . Obviously the functional  $F_k$  is linear. In addition

$$\begin{aligned} |F_k(x)| &= |\xi_k| = \frac{|\xi_k| \|\varphi_k\|}{\|\varphi_k\|} = \frac{\left\| \sum_{i=1}^k \xi_i \varphi_i - \sum_{i=1}^{k-1} \xi_i \varphi_i \right\|}{\|\varphi_k\|} \\ &\leq 2 \sup_n \left\| \sum_{i=1}^n \xi_i \varphi_i \right\| \frac{1}{\|\varphi_k\|} = \frac{2\|y\|}{\|\varphi_k\|} = \frac{2\|A^{-1}x\|}{\|\varphi_k\|} \leq \frac{2\|A^{-1}\|}{\|\varphi_k\|} \|x\|. \end{aligned}$$

Consequently  $\|F_k\| \leq 2\|A^{-1}\|/\|\varphi_k\|$ , i.e., the functionals  $F_k$  are bounded for any  $k$ .

Thus for any  $x \in B$  we have

$$x = \sum_{i=1}^{\infty} F_i(x)\varphi_i = \sum_{i=1}^{\infty} \xi_i\varphi_i.$$

In particular we set  $x = \varphi_j$ . Then  $\xi_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ , by the uniqueness of the expansion of  $x$  in the basis, i.e.,

$$(F_i, \varphi_j) = F_i(\varphi_j) = \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

and  $(F_i, \varphi_j)$  no longer denotes the inner product in a Hilbert space but is merely another notation for the linear functional  $F_i(\varphi_j)$ . This point was discussed in Chapter 2. We call the sequence  $\{F_i\}$  the *conjugate biorthogonal* sequence to the sequence  $\{\varphi_i\}$ . We emphasize that  $F_i$  belongs to  $B^*$ , the dual space, and in general  $B^*$  does not coincide with  $B$ . In Definition 4 a biorthogonal system was defined as a sequence belonging to the same space  $H$  as the original system. The sequence  $\{F_j\}$  belongs to  $B^*$ , and therefore is called conjugate biorthogonal.

In the case of a Hilbert space a sequence biorthogonal to a basis can always be exhibited in the original space. To do this it obviously suffices to prove that the Hilbert space has the property  $H = H^*$ . Indeed the following isomorphism theorem for the spaces  $H$  and  $H^*$  holds.

**THEOREM 4.** *Let  $H$  be a real Hilbert space. For every continuous linear functional  $F$  on  $H$  there exists a unique element  $h_0 \in H$  such that*

$$F(h) = (h, h_0), \quad h \in H,$$

and  $\|F\| = \|h_0\|$ . Conversely, if  $h_0 \in H$ , then  $F(h) = (h, h_0)$  is a continuous linear functional on  $H$  and  $\|F\| = \|h_0\|$ . Thus the spaces  $H$  and  $H^*$  are isomorphic.

**PROOF:** It is obvious that for any vector  $h_0 \in H$  the formula  $F(h) = (h, h_0)$  defines a linear functional. Since  $|F(h)| \leq \|h\|\|h_0\|$ , this functional is continuous and  $\|F\| \leq \|h_0\|$ . For  $h = h_0$  equality is attained in this last inequality:  $F(h_0) = (h_0, h_0) = \|h_0\|^2 = \|h_0\|\|h_0\|$ . Therefore

$$\|F\| = \|h_0\|.$$



We shall now show that every continuous linear functional is representable in the form of the inner product with some vector  $h_0$ . If  $F = 0$ , we set  $h_0 = 0$ . Now suppose  $F \neq 0$  and let  $H_0 = \{h : F(h) = 0\}$  be the linear manifold of zeros of the functional  $F$ . (Since  $F$  is continuous, the linear manifold  $H_0$  is closed.) We remark that now  $H_0 \neq H$ . In this case by the theorem on orthogonal complements there exists a nonzero element  $f_0 \in H \ominus H_0$ . Consider the elements  $F(h)f_0 - F(f_0)h$ , where  $h$  ranges over all of  $H$ . These elements all belong to  $H_0$ . Consequently  $(F(h)f_0 - F(f_0)h, f_0) = 0$ , whence  $F(h)(f_0, f_0) = (h, F(f_0)f_0)$ . If we set  $h_0 = [F(f_0)/(f_0, f_0)]f_0$ , the equality just obtained implies  $F(h) = (h, h_0)$ . This is the required representation of the functional.

It is unique. For suppose otherwise. Then  $(h, h'_0) = (h, h''_0)$  for any vector  $h \in H$  and  $h'_0 \neq h''_0$ . Taking  $h = h'_0 - h''_0$ , we would obtain  $\|h'_0 - h''_0\| = 0$ , i.e.,  $h'_0 = h''_0$ , contrary to the assumption that  $h'_0 \neq h''_0$ . ■

REMARK 1. Theorem 4 holds also in the case of a complex space  $H$ , but the mapping of  $H$  into  $H^*$  becomes a conjugate-isomorphism, i.e., the element  $\lambda h_0$  corresponds to the functional  $\bar{\lambda}F$ .

REMARK 2. Theorem 4 also establishes the general form of a continuous linear functional on  $H$ . To be specific, every continuous linear functional on  $H$  has the form  $F(h) = (h, h_0)$ , where  $h_0$  is a fixed element of the space.

Now let  $\{\varphi_n\}$  be a basis of the Hilbert space  $H$  and let  $g = \sum_{i=1}^{\infty} \xi_i \varphi_i$ . As in the case of a Banach space, we introduce the functional  $F_i$  that assigns to the vector  $g$  the coefficient  $\xi_i$  of its expansion in the basis  $\{\varphi_i\}$ :  $F_i(g) = \xi_i$ .

Using the theorem on the general form of a linear functional on a Hilbert space, we represent the functional  $F_i(g)$  in the form of a scalar product, i.e., we write that  $F_i(g) = (g, g_i)$ , where  $g_i$  is some vector of the space  $H$ .

Choosing as  $g$  the vector  $\varphi_j$ ,  $j = 1, 2, \dots$ , and using the relations that hold for an arbitrary Banach space:  $F_i(\varphi_j) = \delta_{ij}$ , we obtain  $F_i(\varphi_j) = (\varphi_j, g_i) = \delta_{ij}$ , i.e.,  $\{g_i\}$  is the biorthogonal system to  $\{\varphi_j\}$ . Here the notation  $(\varphi_j, g_i)$  now denotes the inner product on  $H$ . The system  $\{g_i\}$  is uniquely determined.

If the vector  $g$  is orthogonal to all  $g_i$  for any  $i = 1, 2, \dots$ , then  $\xi_i = (g, g_i) = 0$  for any  $i$  and consequently, since  $\{\varphi_i\}$  is a basis, we find that  $g = \sum_{i=1}^{\infty} \xi_i \varphi_i = 0$ . Therefore, according to Proposition 1 the sequence biorthogonal to a basis is always complete\* in  $H$ .

Moreover we have the following result.

\*We remark that the system  $\{g_i\}$  is not in general orthonormal, as was

**THEOREM 5 (Banach).** *The sequence  $\{\psi_j\}$  biorthogonal to a basis  $\{\varphi_j\}$  of a Hilbert space  $H$  is also a basis of  $H$ .*

**PROOF:** Any vector  $f \in H$  can be expanded in a norm-convergent series

$$f = \sum_{j=1}^{\infty} (f, \psi_j) \varphi_j,$$

and therefore for any  $h \in H$  the numerical series

$$(f, h) = \sum_{j=1}^{\infty} (f, \psi_j) (\varphi_j, h)$$

converges. Thus  $(f, h) = \lim_{n \rightarrow \infty} \left( f, \sum_{j=1}^n (h, \varphi_j) \psi_j \right)$  for any vector  $f \in H$ . But this means that the sequence

$$h_n = Q_n h = \sum_{j=1}^n (h, \varphi_j) \psi_j, \quad n = 1, 2, \dots, h \in H$$

converges weakly to a vector  $h$ . Indeed, for any continuous linear functional  $F = (\cdot, f)$  we have the equality  $F(h_n) = (h_n, f)$  and  $\overline{F(h_n)} = (f, h_n) \rightarrow (f, h) = \overline{F(h)}$  as  $n \rightarrow \infty$ . But, as was shown in Chapter 2, every weakly convergent sequence is bounded, and therefore for any  $n$  we have  $\|Q_n h\| \leq M \cdot \|h\|$ ,  $\|h\| = 1$ , and consequently  $\|Q_n\| \leq M = \text{const}$ . Thus it has been proved that the norms of the operators  $Q_n$  are uniformly bounded.

By the completeness of the sequence  $\{\psi_j\}$  in  $H$  for every  $\varepsilon > 0$  and any  $h \in H$  there exist numbers  $c_j^\varepsilon$  ( $j = 1, 2, \dots, N_\varepsilon$ ) such that  $\left\| h - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \psi_j \right\| < \varepsilon$ , and therefore for the vector

$$Q_n \left( h - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \psi_j \right) = Q_n h - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \psi_j, \quad n > N_\varepsilon,$$

we have

$$\left\| Q_n h - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \psi_j \right\| \leq M\varepsilon, \quad n > N_\varepsilon.$$

---

required in Proposition 1. However, applying the Gram-Schmidt orthogonalization procedure to the system  $\{g_i\}$ , we find from Proposition 1 that the orthonormal system obtained is complete. Hence, according to the orthogonalization formulas, it follows that the system  $\{g_i\}$  is also complete.



It follows from the inequalities just obtained that for  $n > N_\varepsilon$

$$\|Q_n h - h\| < (1 + M)\varepsilon.$$

Thus any vector  $h \in H$  can be expanded in a norm-convergent series  $h = \sum_{j=1}^{\infty} c_j \psi_j$ , and the coefficients  $c_j$ ,  $j = 1, 2, \dots$ , are uniquely determined from the equalities

$$c_j = (h, \varphi_j), \quad j = 1, 2, \dots \quad \blacksquare$$

DEFINITION 5. The systems  $f_1, f_2, \dots$  and  $g_1, g_2, \dots$  are called *quadratically near* if

$$\sum_{k=1}^{\infty} \|f_k - g_k\|^2 < \infty.$$

THEOREM 6. If  $\varphi_1, \varphi_2, \dots$  is an orthonormal basis in a Hilbert space  $H$  and  $f_1, f_2, \dots$  is an orthonormal system of vectors quadratically near to the basis  $\varphi_1, \varphi_2, \dots$ , then  $f_1, f_2, \dots$  is also an orthonormal basis of the space  $H$ .

PROOF: We shall prove that if the vector  $f_0$  is perpendicular to  $f_i$  for all  $i = 1, 2, \dots$ , then  $f_0 = 0$ . By Proposition 1 of this section, the completeness of the system  $f_1, f_2, \dots$  will follow from this, and by Theorem 1 it will then follow that the system is also a basis.

Suppose it does not follow from the relations  $f_0 \perp f_i$ ,  $i = 1, 2, \dots$  that  $f_0 = 0$ , i.e., suppose  $f_0 \neq 0$ . Then  $f_0, f_1, f_2, \dots$  is an orthogonal set of nonzero vectors, and therefore it is a linearly independent set. We shall show that this cannot be. We choose  $M$  such that  $\sum_{k>M} \|\varphi_k - f_k\|^2 < 1$ . We shall verify that the  $M+1$  vectors  $f_0, f_1, \dots, f_M$  are linearly dependent. Let

$$g_k = \sum_{j=1}^M (f_k, \varphi_j) \varphi_j, \quad k = 0, 1, \dots, M.$$

There are altogether  $M+1$  of these vectors  $g_k$ , and they all belong to the closure of the linear manifold spanned by the vectors  $\varphi_1, \varphi_2, \dots, \varphi_M$ .

Therefore the vectors  $g_0, g_1, \dots, g_M$  are linearly dependent. Let  $\sum_{k=0}^M c_k g_k = 0$ . Then, substituting the expression for  $g_k$  into this equality, we have

$$\sum_{k=0}^M c_k \sum_{j=1}^M (f_k, \varphi_j) \varphi_j = \sum_{j=1}^M \left( \sum_{k=0}^M c_k f_k, \varphi_j \right) \varphi_j = 0.$$

The vectors  $\varphi_j$ ,  $j = 1, 2, \dots, M$ , are linearly independent. Consequently all the coefficients in the last expression are zero, i.e.,  $\left(\sum_{k=0}^M c_k f_k, \varphi_j\right) = 0$ ,  $j = 1, 2, \dots, M$ . Let  $h = \sum_{k=0}^M c_k f_k$ . Then we find that  $h$  is orthogonal to all the vectors  $\varphi_1, \varphi_2, \dots, \varphi_M$ .

We now compute the norm of the vector  $h$ :  $\|h\|^2 = \sum_{k=1}^{\infty} |(h, \varphi_k)|^2 = \sum_{k=M+1}^{\infty} |(h, \varphi_k)|^2$ . On the other hand the vector  $h$  belongs to the closure of the linear manifold spanned by the vectors  $f_0, f_1, \dots, f_M$ , i.e., the vector  $h$  is orthogonal to  $f_{M+1}, f_{M+2}, \dots$ . Therefore

$$\|h\|^2 = \sum_{k=M+1}^{\infty} |(h, \varphi_k) - (h, f_k)|^2 \leq \sum_{k=M+1}^{\infty} \|h\|^2 \|\varphi_k - f_k\|^2 < \|h\|^2.$$

It follows from this that the vector  $h = 0$ , i.e., the vectors  $f_0, f_1, \dots, f_M$  are linearly dependent. ■

### 1.6. The Matrix Representation of a Bounded Linear Operator on $H$

We now pause to study in detail the matrix representation of a bounded operator in an orthonormal basis.

Let  $A$  be a bounded linear operator defined on all of a separable Hilbert space  $H$ . We shall show that it admits a matrix representation analogous to the matrix representation of a linear operator on a finite-dimensional space.

Let  $\{\varphi_n\}$  be an orthonormal basis in  $H$ . We set  $A\varphi_n = g_n$ , and let  $(g_k, \varphi_j) = (A\varphi_k, \varphi_j) = a_{jk}$ ,  $j, k = 1, 2, \dots$ . The numbers  $a_{jk}$  are the Fourier coefficients in the expansion of the vector  $g_k$  in the basis  $\{\varphi_j\}$ . Therefore

$$g_k = \sum_{j=1}^{\infty} a_{jk} \varphi_j, \quad k = 1, 2, \dots$$

and

$$\sum_{j=1}^{\infty} |a_{jk}|^2 = \sum_{j=1}^{\infty} |(g_k, \varphi_j)|^2 < \infty, \quad j, k = 1, 2, \dots$$

We shall show that the matrix  $\{a_{jk}\}_{j,k=1}^{\infty}$  determines the operator  $A$ , i.e., we shall show that from the matrix  $\{a_{jk}\}$  and the orthonormal basis  $\{\varphi_k\}$  the operator can be recovered unambiguously. It has been shown that  $A\varphi_k =$



$g_k = \sum_{j=1}^{\infty} a_{jk} \varphi_j$ ,  $k = 1, 2, \dots$ . The problem is to find the value of  $Af$  for any vector  $f \in H$ . Let  $f = \sum_{k=1}^{\infty} \xi_k \varphi_k$  and  $f_N = \sum_{k=1}^N \xi_k \varphi_k$ . Then  $Af_N = \sum_{k=1}^{\infty} \eta_k^N \varphi_k$ , where  $\eta_k^N = \sum_{j=1}^N a_{kj} \xi_j$ . We have further, by the continuity of the operator  $A$ , that

$$c_k = (Af, \varphi_k) = \lim_{N \rightarrow \infty} (Af_N, \varphi_k) = \lim_{N \rightarrow \infty} \eta_k^N = \lim_{N \rightarrow \infty} \sum_{j=1}^N a_{kj} \xi_j = \sum_{j=1}^{\infty} a_{kj} \xi_j.$$

Consequently for any vector  $f \in H$  we have the equality  $Af = \sum_{k=1}^{\infty} c_k \varphi_k$ , where  $c_k = \sum_{j=1}^{\infty} a_{kj} \xi_j$ , i.e., we have indeed obtained a matrix representation of the operator  $A$  in the basis  $\{\varphi_k\}$ .

## EXAMPLES

### 1.6.1. Strictly Convex Sets.

If the point  $h$  of a Hilbert space  $H$  belongs to the open interval joining the points  $f$  and  $g$  (the closed interval joining the points  $f$  and  $g$  is the set of vectors of the form  $tf + (1-t)g$  for  $0 \leq t \leq 1$ ), then  $h$  is representable in the form  $h = tf + (1-t)g$ ,  $0 < t < 1$ , and  $h$  is called an *interior point* of the closed interval. If a point of a convex set is not an interior point of any closed interval contained in the set, then the point is called an *extreme point* of the set. A closed convex set in  $H$  is called *strictly convex* if all of its boundary points are extreme points.

We shall show, for example, that the unit ball in any Hilbert space is a strictly convex set. The boundary points of the unit ball are the vectors  $f$  for which  $\|f\| = 1$ . Therefore we need to show that if  $f = tg + (1-t)h$ , where  $0 < t < 1$ ,  $\|f\| = 1$ ,  $\|g\| \leq 1$ , and  $\|h\| \leq 1$ , then  $f = g = h$ . We have  $1 = (f, f) = (f, tg + (1-t)h) = t(f, g) + (1-t)(f, h)$ . But  $|(f, g)| \leq 1$  and  $|(f, h)| \leq 1$ , and by the strict convexity of the closed unit disk in the plane, we have  $(f, g) = (f, h) = 1$ . But then the Cauchy-Bunyakovskii inequality becomes an equality. Consequently the vectors  $g$  and  $f$  and the vectors  $h$  and  $f$  are collinear, i.e.,  $g = \alpha f$ ,  $h = \beta f$ , where  $\alpha$  and  $\beta$  are scalars. But then  $1 = (f, g) = (f, \alpha f) = \alpha$ , and  $1 = (f, h) = (f, \beta f) = \beta$ , i.e.,  $f = g = h$ .

### 1.6.2. Strong and Weak Convergence in $H$ .

A Hilbert space is a metric space and consequently also a topological space. The strong topology, as we know, is given by a system of neighbor-

hoods of zero of the form  $\Sigma_{0,\varepsilon} = \{f : \|f\| < \varepsilon\}$ , where  $\varepsilon$  is an arbitrary positive number.

The weak topology is given by a system of neighborhoods of zero of the form  $\Sigma_{\varepsilon,n} = \{f : |F_i(f)| < \varepsilon, i = 1, 2, \dots, n\}$ , where  $F_i, i = 1, 2, \dots, n$ , are continuous linear functionals on  $H$ . Taking account of Theorem 4 on the general form of a functional in  $H$ , we see that to define the weak topology it suffices to prescribe a system of neighborhoods of zero of the form  $\Sigma_{0,n} = \{f : |(f, h_i)| < \varepsilon, i = 1, 2, \dots, n\}$ .

We shall show that each set that is closed in the weak topology is also closed in the strong topology.

Let  $M$  be a set that is closed in the weak topology in  $H$  and  $f_n \in M$ ,  $\|f_n - f\| \rightarrow 0$ . We shall prove that  $f \in M$ . To do this we remark that strong convergence implies weak convergence. In fact we have the estimate

$$|(f_n, g) - (f, g)| \leq \|f_n - f\| \cdot \|g\| \rightarrow 0.$$

Thus if  $\{f_n\}$  converges strongly to the vector  $f$ , i.e.,  $\|f_n - f\| \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\{f_n\}$  converges weakly to  $f$ . Since  $M$  is closed in the weak topology, i.e., with respect to weak convergence, and  $f_n \in M$ , we find that  $f \in M$ , which was to be proved.

We note that strong convergence naturally does not follow from weak convergence. In fact, let  $\{\varphi_n\}$  be an orthonormal sequence. Then for any vector  $f \in H$  the Fourier coefficient  $(f, \varphi_n)$  over this orthonormal system tends to zero because the series  $\sum_{n=1}^{\infty} |(f, \varphi_n)|^2$  converges:  $(f, \varphi_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Consequently  $\varphi_n$  converges weakly to zero. However, since  $\|\varphi_n\| = 1$  for any  $n$ , the sequence  $\{\varphi_n\}$  does not converge strongly to zero. Hence in particular it follows that the sequence  $\{\varphi_n\}$  is not closed in the weak topology, while in the strong topology it is discrete, having no limit points (since this sequence is orthonormal) and therefore closed.

We shall prove another proposition that often turns out to be useful. To be specific, if the sequence  $\{f_n\}$  converges to the vector  $f$  weakly and  $\|f_n\| \rightarrow \|f\|$ , then  $\|f_n - f\| \rightarrow 0$ , i.e.,  $f_n \rightarrow f$  strongly. Indeed,

$$\|f_n - f\|^2 = (f_n - f, f_n - f) = \|f_n\|^2 - (f, f_n) - (f_n, f) + \|f\|^2.$$

By the weak convergence of  $f_n$  to  $f$  we have  $(f, f_n) \rightarrow \|f\|^2$  and  $(f_n, f) \rightarrow \|f\|^2$ . By hypothesis  $\|f_n\| \rightarrow \|f\|$ , so that  $\|f_n - f\| \rightarrow 0$  as  $n \rightarrow \infty$ , which was to be proved.

### 1.6.3. The Space $A^2(D)$ .

Let  $D = \{z : |z| < 1\}$  be the open unit disk in the complex plane  $\mathbb{C}$ , and  $A^2(D)$  the set of functions that are analytic and square-integrable in  $D$ .



Then  $A^2(D)$  is a vector space with the usual function operations. On this space we can introduce an inner product by setting  $(f, g) = \int_D f(z) \overline{g(z)} dz$ .

It can be shown that this space is complete in the metric just introduced and hence is a Hilbert space. A simple verification shows that the functions

$\varphi_n(z) = \sqrt{\frac{n+1}{\pi}} z^n$  form an orthonormal system. Indeed

$$\begin{aligned} & \frac{(m+1)(n+1)}{\pi} \int_{|z|<r} z^n \bar{z}^m dz \\ &= \frac{(m+1)(n+1)}{\pi} \int_0^{2\pi} \int_0^r e^{i(n-m)\theta} \rho^{n+m+1} d\rho d\theta \\ &= \frac{2(m+1)(n+1)}{(m+n+2)} r^{m+n+2} \delta_{n,m}, \end{aligned}$$

where

$$\delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

Thus setting  $r = 1$ , we obtain  $(\varphi_n, \varphi_m) = \delta_{n,m}$ . In fact the vectors  $\varphi_n(z)$  form a complete system, i.e., a basis in the space  $A^2(D)$ .

Indeed, let the function  $f(z) \in A^2(D)$  have the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n = \sum_{n=0}^{\infty} \left( \frac{\pi}{n+1} \right)^{1/2} \alpha_n \varphi_n(z).$$

By the uniform convergence of the Taylor series we have  $(f, \varphi_n) = \left( \frac{\pi}{n+1} \right)^{1/2} \alpha_n$ . According to Bessel's inequality the series  $\sum_{n=0}^{\infty} \frac{\pi}{n+1} |\alpha_n|^2$  converges. Therefore the series  $\sum_{n=0}^{\infty} \left( \frac{\pi}{n+1} \right)^{1/2} \alpha_n \varphi_n(z) = f(z)$  converges in the norm of the space  $A^2(D)$ .

Thus we have given above two different proofs of the fact that the system  $\{\varphi_n\}$  is a basis of the space  $A^2(D)$ . On the one hand, assuming that  $f \perp \varphi_n$ , we found by the formula  $(f, \varphi_n) = \left( \frac{\pi}{n+1} \right)^{1/2} \alpha_n$  that all  $\alpha_n = 0$ , i.e., that the function  $f(z) \equiv 0$ , i.e., the system  $\{\varphi_n\}$  is complete, and therefore a basis. On the other hand, it was shown that every function  $f$  can be expanded in a norm-convergent series in the system  $\{\varphi_n\}$ , and moreover in a unique way, i.e., the system  $\{\varphi_n\}$  is a basis of the space  $A^2(D)$ .

1.6.4. AN EXAMPLE OF A DENSE SET IN  $l^2$ .

Let  $0 < |\alpha| < 1$ . It happens that the closure of the linear manifold spanned by the set of vectors of the form

$$g_k = (1, \alpha^k, \alpha^{2k}, \dots), \quad k = 1, 2, \dots$$

coincides with the whole space  $l^2$ .

Indeed, let the vector  $f = (\xi_0, \xi_1, \dots)$  be orthogonal to all the vectors  $g_k$ . We have

$$0 = (f, g_k) = \sum_{n=0}^{\infty} \xi_n \bar{\alpha}^{nk} = \sum_{n=0}^{\infty} \xi_n z_k^n,$$

where  $z_k = \bar{\alpha}^k$ ,  $k = 1, 2, \dots$ .

Therefore, as can easily be concluded from the form of the coefficients in the Taylor series, the function  $\sum \xi_n z^n$ , which is analytic in the disk  $|z| < 1$ , has infinitely many zeros  $z_k$  and  $z_k \rightarrow 0$  as  $k \rightarrow \infty$ .

Such a function is identically zero; consequently all the coefficients  $\xi_n$  are 0 and therefore the vector  $f = 0$ , which was to be shown.

1.6.5. A Countably Additive Measure on  $H$ .

We introduce a countably additive nonnegative function  $\mu$  on the Hilbert space  $H$ . We require that  $\mu(\Sigma) > 0$  for any nonempty open set  $\Sigma \subset H$  and that  $\mu(f + \Sigma) = \mu(\Sigma)$  for any vector  $f$  and any open set  $\Sigma$ . It turns out that for any such countably additive function and any nonempty ball  $B$  the relation  $\mu(B) = \infty$  holds. In other words, if the function is called a measure on  $H$ , then the measure of any nonempty ball in  $H$  is infinite.

Let  $\{\varphi_k\}$  be an orthonormal system in  $H$ . Let  $B_n = \left\{ f : \left\| f - \frac{r}{2}\varphi_n \right\| < \frac{r}{4}, r > 0 \right\}$ .

Thus the sets  $B_n$  are balls with centers at the points  $\frac{r}{2}\varphi_n$  and radii  $r/4$ . If  $f \in B_n$ , then

$$\|f\| \leq \left\| f - \frac{r}{2}\varphi_n \right\| + \left\| \frac{r}{2}\varphi_n \right\| < r,$$

i.e.,  $f \in B$ , so that  $B_n \subset B$ . If  $f \in B_n$  and  $g \in B_m$ , then for  $n \neq m$  we have  $\|f - g\| > 0$ , i.e., the balls  $B_n$  and  $B_m$  are disjoint. Indeed,

$$\left\| \frac{r}{2}\varphi_n - \frac{r}{2}\varphi_m \right\| \leq \left\| \frac{r}{2}\varphi_n - f \right\| + \|f - g\| + \left\| g - \frac{r}{2}\varphi_m \right\|,$$

$$\|f - g\| \geq r \frac{\sqrt{2}}{2} - \frac{r}{4} - \frac{r}{4} > 0.$$



Thus the ball  $B$  contains infinitely many disjoint open balls of the same positive measure, i.e.

$$\mu(B) \geq \sum_{n=1}^{\infty} \mu(B_n) = \infty.$$

#### 1.6.6. A BASIS OF TRIGONOMETRIC FUNCTIONS.

In the space  $L^2[0, 2\pi]$  the trigonometric system of functions  $\left\{ \frac{1}{\sqrt{2\pi}} e^{int} \right\}$ ,  $n = 0 \pm 1, \dots$  forms a basis of the space and consequently every vector  $f \in L^2[0, 2\pi]$  can be expanded in a norm-convergent series

$$f = \sum_{-\infty}^{\infty} c_n \varphi_n, \quad \varphi_n = \frac{1}{\sqrt{2\pi}} e^{int}, \quad c_n = (f, \varphi_n) = \int_0^{2\pi} f(t) \overline{\varphi_n(t)} dt.$$

We shall now prove that the system  $\{\varphi_n\}$  is complete. According to Theorem 1 it will then follow that it forms a basis, since it is obvious that it is orthonormal. Suppose there exists a nonzero function  $g \in L^2[0, 2\pi]$  such that

$$\int_0^{2\pi} g(t) e^{-int} dt = 0, \quad n = 0, \pm 1, \dots,$$

Integrating these relations by parts, we find that for any constant  $c$  we have the relations

$$\int_0^{2\pi} \{F(t) - c\} e^{-int} dt = 0, \quad F(t) = \int_0^t g(\xi) d\xi, \quad n = \pm 1, \pm 2, \dots$$

We choose the constant  $c$  so that this equality holds for  $n = 0$  also, i.e., we set

$$c = \frac{1}{2\pi} \int_0^{2\pi} F(t) dt. \text{ By the Weierstrass theorem for any } \varepsilon > 0 \text{ there exists}$$

a trigonometric polynomial  $\sigma(t) = \sum_{k=-N}^N A_k e^{ikt}$  such that  $|\Phi(t) - \sigma(t)| < \varepsilon$ , where  $\Phi(t) = F(t) - c$ . Therefore

$$\begin{aligned} \int_0^{2\pi} \Phi(t) \overline{\Phi(t)} dt &= \int_0^{2\pi} |\Phi(t)|^2 dt = \int_0^{2\pi} \overline{\Phi(t)} [\Phi(t) - \sigma(t)] dt \\ &\leq \varepsilon \int_0^{2\pi} |\Phi(t)| dt \leq \varepsilon (2\pi)^{1/2} \left( \int_0^{2\pi} |\Phi(t)|^2 dt \right)^{1/2}. \end{aligned}$$

Thus

$$\int_0^{2\pi} |\Phi(t)|^2 dt \leq 2\pi \varepsilon^2.$$

Since  $\varepsilon$  is arbitrary, it follows that  $\Phi(t) \equiv 0$ , i.e.,  $F(t) \equiv \text{const.}$  Consequently  $g(t) = 0$  almost everywhere (cf. Sec. 3.4).

### EXERCISES

1. The polynomials obtained by orthogonalization in the space  $L^2[-1, 1]$  starting with the functions  $1, x, x^2, \dots$  are called the *Legendre polynomials*. Show that the  $n$ th Legendre polynomial has the form

$$P_n(x) = c_n[(x^2 - 1)^n]^{(n)}.$$

2. Prove that the Legendre polynomials are complete in the space  $L^2[-1, 1]$ .

3. In the set of functions satisfying the condition  $\int_{-1}^1 \frac{x^2(t)}{\sqrt{1-t^2}} dt < \infty$ , we define an inner product by the formula

$$(x, y) = \int_{-1}^1 \frac{x(t)y(t)}{\sqrt{1-t^2}} dt.$$

Show that orthogonalizing the system of functions  $x_n(t) = t^n$ ,  $n = 0, 1, \dots$  with respect to this inner product leads (up to a constant) to the polynomials  $T_n(t) = \cos(n \arccos t)$ ,  $n = 1, 2, \dots$  (the *Chebyshev polynomials*).

4. The functions obtained by orthogonalization of the system  $t^n e^{-t}$ ,  $n = 0, 1, \dots$ , in the space  $L^2[0, \infty)$  are called the *Laguerre functions*. Show that the  $n$ th Laguerre function has the form

$$L_n(t) = c_n e^t \frac{d^n}{dt^n} (t^n e^{-2t}).$$

5. The system of Haar functions  $\chi_k$ ,  $\chi_k \in \bar{C}^2[0, 1]$ , (where  $\bar{C}^2[0, 1]$  is the space of piecewise continuous functions whose values at points of discontinuity are the average of their right- and left-hand limits at those points with metric given by the rule  $\|x - y\|^2 = (x - y, x - y) = \rho^2(x, y) = \int_0^1 [x(t) - y(t)]^2 dt$ ) is defined as follows:

$$\chi_1 = 1, \quad \chi_2 = \begin{cases} 1, & 0 < t < \frac{1}{2}, \\ -1, & \frac{1}{2} < t < 1; \end{cases}$$

$$\chi_{2^k+s} = \begin{cases} \sqrt{2^k}, & t \in \left(\frac{s-1}{2^k}, \frac{2s-1}{2^{k+1}}\right), \\ -\sqrt{2^k}, & t \in \left(\frac{2s-1}{2^{k+1}}, \frac{s}{2^k}\right), \\ 0, & t \notin \left(\frac{s-1}{2^k}, \frac{s}{2^k}\right). \end{cases} \quad 1 \leq s \leq 2^k, k = 1, 2, \dots$$



Prove that this system is orthonormal and each function in  $C[0, 1]$  can be uniformly approximated by polynomials in the Haar system.

6. Construct a system of continuous functions  $a_1(x), a_2(x), \dots$ , that will be orthogonal in the space  $L^2[a, b]$  and will have the following properties:

a) the relation  $\int_a^b f(x) a_n(x) dx = 0$  for  $n = 1, 2, \dots$ , implies that  $f(x) \equiv 0$  for any continuous function  $f(x)$ ;

b) the linear combinations of the functions  $a_1(x), a_2(x), \dots$  are not dense in the space  $L^2[a, b]$ .

7. Let  $\{f_k\}$  be a complete system of vectors in  $H$ . Let  $\lambda_{1n}^{(1)}, \lambda_{1n}^{(n)}$  be the smallest and largest eigenvalues of the Gram matrix  $\{\alpha_{jk}\}_{j,k=1}^n$ ,  $\alpha_{jk} = (f_k, f_j)$ . If  $\lim_{k \rightarrow \infty} \lambda_{1k}^{(1)} = A > 0$ ,  $\overline{\lim}_{k \rightarrow \infty} \lambda_{1k}^{(n)} = B < \infty$ , then the sequence  $\{f_k\}$  is a basis in  $H$ .

8. Prove that every bounded invertible linear operator  $A$  maps any orthonormal basis of the space  $H$  to another basis of the space  $H$ . The latter is called a *Riesz basis*. If a sequence  $\{\psi_j\}$  is a Riesz basis, then the sequence  $\{\varphi_j\} = \{\psi_j / \|\psi_j\|\}$  is also a Riesz basis.

9. If the sequence  $\{\psi_j\}$  is a Riesz basis, there exist numbers  $a_1 > 0$ ,  $a_2 > 0$ , such that for any  $n$  and any numbers  $\gamma_1, \gamma_2, \dots, \gamma_n$

$$a_2 \sum_{j=1}^n |\gamma_j|^2 \leq \left\| \sum_{j=1}^n \gamma_j \psi_j \right\|^2 \leq a_1 \sum_{j=1}^n |\gamma_j|^2.$$

10. If the sequence  $\{\psi_j\}$  is a Riesz basis, then for any  $f \in H$

$$\sum_{j=1}^{\infty} |(f, \psi_j)|^2 < \infty, \quad \sum_{j=1}^{\infty} |(f, g_j)|^2 < \infty,$$

where  $\{g_j\}$  is the sequence biorthogonal to  $\{\psi_j\}$ .

11. A basis of the space  $H$  is called *permutative* if it remains a basis of  $H$  under any permutation of its elements. Every orthonormal basis is permutative. Moreover any Riesz basis is permutative.

12. A sequence of vectors  $g_j$  is called  $\omega$ -linearly independent if the equality  $\sum_{j=1}^{\infty} c_j g_j = 0$  is impossible when  $0 < \sum_{j=1}^{\infty} |c_j|^2 \|g_j\|^2 < \infty$ . Prove that

every  $\omega$ -linearly independent sequence  $\{g_j\}$  that is quadratically near to a Riesz basis is a Riesz basis.

13. Prove that if a linear operator  $A$  defined everywhere on a separable Hilbert space  $H$  admits a matrix representation in some orthonormal basis, then it is bounded.

## 2. SPECTRAL THEOREMS

Before proceeding to the discussion of spectral theorems we present several important concepts related to operators.

We emphasize that throughout this section, except in Sec. 4.2.12 we are considering bounded linear operators defined on the whole space.

### 2.1. The Adjoint Operator

DEFINITION 1. An operator  $A^*$  is called the *adjoint* of the bounded linear operator  $A$  if for all  $f, g \in H$  the equality

$$(Af, g) = (f, A^*g)$$

holds.

For fixed  $g$  the function  $(Af, g)$  is a linear functional applied to the variable element  $f$ ; by Theorem 4 of Sec. 4.1 there exists a unique element  $g^*$  such that  $(Af, g) = (f, g^*)$  for any  $f$ . We define  $A^*g = g^*$ ; the operator so defined is obviously linear. We shall show that it is bounded and that its norm is the norm of the operator  $A$ . Let  $f = A^*g$ . Then we can write

$$(A^*g, A^*g) = (AA^*g, g) \leq \|AA^*g\| \|g\| \leq \|A\| \|A^*g\| \|g\|.$$

Therefore  $\|A^*g\| \leq \|A\| \|g\|$ ,  $\|A^*\| \leq \|A\|$ . In exactly the same way, setting  $g = Af$ , we find that  $\|A\| \leq \|A^*\|$ . Consequently  $\|A\| = \|A^*\|$ .

The identity operator  $E$  and the zero operator  $O$  coincide with their adjoints, i.e.,  $E = E^*$  and  $O^* = O$ .

DEFINITION 2. If a bounded linear operator coincides with its adjoint, it is called *symmetric* (or *self-adjoint*).

The following equalities are consequences of the definition of the adjoint operator:

$$(aA)^* = \bar{a}A^*, \quad (A_1 + A_2)^* = A_1^* + A_2^*, \quad (A_1 A_2)^* = A_2^* A_1^*, \quad (A^*)^* = A.$$

If the sequence  $\{A_n\}$  converges in norm to  $A$ , then by the fact that  $\|A^*\| = \|A\|$ , it follows that the sequence  $\{A_n^*\}$  converges in norm to  $A^*$ :  $A_n^* \Rightarrow A^*$  as  $n \rightarrow \infty$ .



The adjoint operator can also be defined for more general spaces than Hilbert spaces. Suppose given, for example, two normed vector spaces  $N_1$  and  $N_2$  and an operator  $A : N_1 \rightarrow N_2$ . Let  $\Phi(g)$  be a linear functional on  $N_2$ . If  $g = Af$ ,  $f \in N_1$ ,  $g \in N_2$ , then  $\Phi(g) = \Phi(Af) = F(f)$ , where  $F$  is a functional defined on  $N_1$ . It is obvious that the functional  $F$  is linear. Thus we find that to each functional  $\Phi$  of  $N_2^*$  a functional  $F$  of  $N_1^*$  is assigned, i.e., we have constructed an operator  $A^* : N_2^* \rightarrow N_1^*$ . This operator  $A^*$  is called the *adjoint* to the operator  $A$  and the equality  $\Phi(g) = F(f)$  can be written in the form  $F = A^* \Phi$ . If we recall that in normed (or vector spaces) linear functionals can be written in a form similar to that used for the functionals in a Hilbert space, i.e., using an analog of the inner product, then from what has just been said we obtain a notation completely analogous to that used in Hilbert spaces:

$$\Phi(g) = (g, \Phi) = (Af, \Phi) = F(f) = (f, F) = (f, A^* \Phi).$$

Here  $\Phi(g) = (g, \Phi) = (\Phi, g)$  is a notation for the functional in the form analogous to an inner product (cf. Sec. 2.2.5). In exactly the same way we obtain  $F(f) = (f, F) = (F, f)$ . In particular the elements  $g$  and  $\Phi$  are called *orthogonal* (in a normed space!) if  $\Phi(g) = (g, \Phi) = (\Phi, g) = 0$ .

All the properties of the adjoint operator that hold for Hilbert space carry over to normed spaces.

We shall say that a sequence of operators  $\{A_n\}$  on Hilbert space *converges weakly* to the operator  $A$  if for any  $f, g \in H$

$$(A_n f, g) \rightarrow (A f, g).$$

It is obvious that if  $\{A_n\}$  converges weakly to  $A$ , then  $\{A_n^*\}$  converges weakly to  $A^*$  as  $n \rightarrow \infty$ .

On the other hand pointwise convergence  $A_n \rightarrow A$  in general does not imply pointwise convergence  $A_n^* \rightarrow A^*$  as  $n \rightarrow \infty$ . Indeed, suppose we are given operators

$$A_n(x_1, x_2, \dots) = (x_{n+1}, x_{n+2}, \dots), \quad x = (x_1, x_2, \dots) \in l^2,$$

on  $l^2$ . Then

$$A_n^*(x_1, x_2, \dots) = (0, \dots, 0, x_1, x_2, \dots),$$

where there are  $n$  zeros before  $x_1$ . Here  $A_n x \rightarrow 0$  as  $n \rightarrow \infty$  for any  $x$ , while  $\|A_n^* x\| = \|x\|$ . If an inverse operator  $A^{-1}$  exists for the operator  $A$ , then the equalities  $A^*(A^{-1})^* = (A^{-1}A)^* = E^* = E$  hold. Consequently the operator  $A^*$  also has an inverse and  $(A^*)^{-1} = (A^{-1})^*$ .

## 2.2. The Concept of a Completely Continuous Operator

Hilbert was the first to call attention to an important class of operators that can be approximated by finite-dimensional operators, namely the completely continuous operators.

**DEFINITION 3.** A linear operator defined everywhere on  $H$  is *completely continuous* if it maps every bounded set of points into a set having the property that from every infinite sequence of points, one can extract a subsequence that converges to some element of  $H$  (in the sense of the metric of  $H$ ).\*

A completely continuous operator is bounded. Indeed otherwise there would exist a sequence of vectors  $\{f_k\}$ ,  $k = 1, 2, \dots$ , for which  $\|f_k\| = 1$  and  $\|Af_k\| > k$ . But no convergent subsequence can be extracted from the set of points  $\{Af_k\}$ , i.e., we have reached a contradiction.

The following propositions are easily proved.

**PROPOSITION 1.** If  $A$  is completely continuous, and the operator  $B$  is defined everywhere in  $H$  and is bounded, then  $AB$  is completely continuous.

**PROPOSITION 2.** If  $A_1$  and  $A_2$  are completely continuous operators, then  $\alpha_1 A_1 + \alpha_2 A_2$  is also a completely continuous operator.

**THEOREM 1.** If  $A$  is a bounded linear operator defined everywhere in  $H$  and  $A^*A$  is completely continuous, then the operator  $A$  is also completely continuous.

**PROOF:** Let  $M$  be a bounded infinite set of points  $f$  with  $\|f\| < c$ . Let  $\{f_k\}$  be some sequence of elements of this set which is mapped by the operator  $A^*A$  into a convergent sequence. Since

$$\begin{aligned}\|Af_n - Af_m\|^2 &= (A(f_n - f_m), A(f_n - f_m)) \\ &= (A^*A(f_n - f_m), f_n - f_m) \leq \|A^*Af_n - A^*Af_m\| \cdot \|f_n - f_m\|,\end{aligned}$$

and

$$\lim_{n,m \rightarrow \infty} \|A^*Af_n - A^*Af_m\| = 0, \quad \|f_n - f_m\| \leq 2c,$$

it follows that  $\lim_{n,m \rightarrow \infty} \|Af_n - Af_m\| = 0$ , i.e.,  $\{Af_n\}$  converges. ■

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\*It can be shown that completely continuous operators (mappings) are those that map the closed unit ball to a compact set. It is also easy to verify that the closure of the image of a bounded set under a completely continuous mapping is a compact set. Such operators are frequently called *compact*.



**COROLLARY 1.** *If the operator  $A$  is completely continuous, then the operator  $A^*$  has the same property.*

Indeed, if the operator  $A$  is completely continuous, then the operator  $AA^* = (A^*)^* A^*$  is also; one then has only to apply the theorem.

**COROLLARY 2.** *If  $A$  is a completely continuous operator and the linear operator  $B$  defined everywhere in  $H$  is bounded, then the operator  $BA$  is completely continuous.*

Indeed, the operator  $(BA)^* = A^* B^*$  is completely continuous and so the operator  $BA$  is also.

**THEOREM 2.** *An operator  $A$  that is the limit (in the sense of norm convergence in the space of operators) of a sequence of completely continuous operators is itself completely continuous.*

**PROOF:** Consider a sequence of positive numbers  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots$ , with  $\varepsilon_1 > \varepsilon_2 > \dots$ , and  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then there exists a sequence  $A_{\varepsilon_i}$  of completely continuous operators such that  $\|A - A_{\varepsilon_i}\| < \varepsilon_i$ ,  $i = 1, 2, \dots$ . Let  $M$  be an arbitrary bounded set of points  $f \in H$  with  $\|f\| \leq c$ . Let  $\{f_k\}_1^\infty$  be an arbitrary infinite sequence in  $M$ . We extract a subsequence of this sequence  $\{f_{1k}\}_1^\infty$  that the operator  $A_{\varepsilon_1}$  maps to a convergent sequence. From this subsequence we extract a second subsequence  $\{f_{2k}\}_1^\infty$  that the operator  $A_{\varepsilon_2}$  maps to a convergent sequence. We continue this process indefinitely. Now consider the diagonal sequence  $\{f_{kk}\}_1^\infty$ . We shall show that this sequence is mapped into a convergent sequence by the operator  $A$ . We first of all remark that it is mapped into a convergent sequence by each of the operators  $A_{\varepsilon_i}$ . Then

$$\begin{aligned} \|Af_{nn} - Af_{mm}\| &\leq \|(A - A_{\varepsilon_k})f_{nn}\| + \|(A - A_{\varepsilon_k})f_{mm}\| \\ &\quad + \|A_{\varepsilon_k}f_{nn} - A_{\varepsilon_k}f_{mm}\| \leq 2c\varepsilon_k + \|A_{\varepsilon_k}f_{nn} - A_{\varepsilon_k}f_{mm}\|. \end{aligned}$$

By first choosing a sufficiently large index  $k$ , and then indices  $n$  and  $m$  we can make the right-hand side as small as desired. Consequently the sequence  $\{Af_{nn}\}$  converges in  $H$ . ■

### 2.3. The Absolute Norm of an Operator

We consider the concept of the absolute norm of an operator. Let  $H$  be a separable Hilbert space and  $A$  a bounded linear operator defined on all of  $H$ . Let  $\{f_k\}_1^\infty$  and  $\{\varphi_k\}_1^\infty$  be two arbitrary orthonormal bases in  $H$ . We assume that

$$\sum_{i,k=1}^{\infty} |(Af_k, \varphi_i)|^2 < \infty.$$

The class of operators for which this inequality holds is called the *Schmidt class*. Since  $(Af_k, \varphi_i)$ ,  $i = 1, 2, \dots$ , are the Fourier coefficients of the vector  $Af_k$  in the basis  $\{\varphi_i\}_1^\infty$ , Parseval's equality gives

$$\sum_{i,k=1}^{\infty} |(Af_k, \varphi_i)|^2 = \sum_{k=1}^{\infty} \|Af_k\|^2.$$

On the other hand  $(A^*\varphi_i, f_k) = (\varphi_i, Af_k)$  are the Fourier coefficients of the vector  $A^*\varphi_i$  in the basis  $\{f_k\}_1^\infty$ . Therefore

$$\sum_{i,k=1}^{\infty} |(Af_k, \varphi_i)|^2 = \sum_{i=1}^{\infty} \|A^*\varphi_i\|^2.$$

Therefore the quantity  $\|A\|_2 = \sqrt{\sum_{i,k=1}^{\infty} |(Af_k, \varphi_i)|^2}$  is independent of the choice of the bases  $\{f_k\}$  and  $\{\varphi_i\}$ , and depends only on the operator  $A$ . This quantity  $\|A\|_2$  is called the *absolute norm\** of the operator  $A$ .

Since any unit vector can be taken as  $f_1$ , we have

$$\|Af_1\| \leq \|A\|_2, \quad \|A\| \leq \|A\|_2,$$

that is, the ordinary norm does not exceed the absolute norm. It is easy to see that if  $C$  is an arbitrary bounded operator, then

$$\|CA\|_2 \leq \|C\| \|A\|_2, \quad \|AC\|_2 \leq \|C\| \|A\|_2.$$

Furthermore

$$\begin{aligned} \|A + B\|_2 &= \sqrt{\sum_{j=1}^{\infty} \|Af_j + Bf_j\|^2} \leq \sqrt{\sum_{j=1}^{\infty} \{\|Af_j\| + \|Bf_j\|\}^2} \\ &\leq \sqrt{\sum_{j=1}^{\infty} \|Af_j\|^2} + \sqrt{\sum_{j=1}^{\infty} \|Bf_j\|^2} = \|A\|_2 + \|B\|_2, \end{aligned}$$

for any linear operators  $A$  and  $B$ .

Let us set  $f_k = \varphi_k$  in the formulas above for each index  $k$ . Then

$$\|A\|_2 = \sqrt{\sum_{i,k} |(A\varphi_k, \varphi_i)|^2} = \sqrt{\sum_{i,k=1}^{\infty} |a_{ik}|^2},$$

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\*The quantity  $\|A\|_2$  is also called the *Schmidt norm*.



where  $a_{ik} = (A\varphi_k, \varphi_i)$ . Thus if  $\|A\|_2 < \infty$ , then the operator  $A$  admits a matrix representation and

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 < \infty.$$

Moreover the following result holds.

**THEOREM 3.** *If  $\|A\|_2 < \infty$ , i.e., the operator belongs to the Schmidt class, then the operator  $A$  is completely continuous.*

**PROOF:** Let  $\{g_k\}_1^\infty$  be some orthonormal basis of the space  $H$ . Then  $\|A\|_2 = \sqrt{\sum_{k=1}^{\infty} \|A^* g_k\|^2}$ . Let  $\varepsilon > 0$ , and choose the number  $M$  sufficiently large that  $\sum_{k=M+1}^{\infty} \|A^* g_k\|^2 < \varepsilon^2$ . Let  $A_\varepsilon$  be the operator defined by the formula

$$A_\varepsilon f = \sum_{k=1}^M (Af, g_k) g_k.$$

The operator  $A_\varepsilon$  is finite-dimensional and therefore completely continuous. For any  $f \in H$  we have

$$\begin{aligned} \|Af - A_\varepsilon f\|^2 &= \sum_{k=M+1}^{\infty} |(Af, g_k)|^2 = \sum_{k=M+1}^{\infty} |(f, A^* g_k)|^2 \\ &\leq \|f\|^2 \sum_{k=M+1}^{\infty} \|A^* g_k\|^2 \leq \varepsilon^2 \|f\|^2. \end{aligned}$$

Consequently  $\|A - A_\varepsilon\| < \varepsilon$ , and the operator  $A$ , being the limit (in the operator norm) of completely continuous operators, is itself completely continuous. ■

Let us now consider a specific example of an operator of the Schmidt class—the *Hilbert-Schmidt integral operator*.

Let  $\varphi$  be a function of the space  $L^2[a, b]$  and let the function  $K(s, t)$  belong to  $L^2([a, b] \times [a, b])$ , i.e.,  $\int_a^b \int_a^b |K^2(s, t)| ds dt < \infty$ . We define the operator  $A$  by the following rule:

$$A\varphi = \int_a^b K(s, t) \varphi(t) dt.$$

We show first of all that this formula does indeed define an operator on the Hilbert space  $L^2[a, b]$ .

We first remark that by Fubini's theorem the integral  $\int_a^b |K^2(s, t)| dt$  exists for almost all  $s$ , i.e., as a function of  $t$  the kernel  $K(s, t)$  belongs to  $L^2[a, b]$  for almost all  $s$ . Since the product of square-integrable functions is integrable, it follows that  $A\varphi$  exists for almost all  $s \in [a, b]$ . We shall show that  $A\varphi \in L^2[a, b]$ . By the Cauchy-Bunyakovskii inequality we have for almost all  $s$

$$\begin{aligned} |A\varphi|^2 &= \left| \int_a^b K(s, t)\varphi(t) dt \right|^2 \leq \int_a^b |K^2(s, t)| dt \cdot \int_a^b |\varphi^2(t)| dt \\ &= \|\varphi\|^2 \int_a^b |K^2(s, t)| dt. \end{aligned}$$

Integrating on  $s$  and replacing the iterated integral by a double integral, we find

$$\|A\varphi\|^2 \leq \|\varphi\|^2 \int_a^b \int_a^b |K^2(s, t)| ds dt,$$

i.e.,

$$\|A\| \leq \int_a^b \int_a^b |K^2(s, t)| ds dt,$$

and the operator  $A$  is indeed bounded.

We now show that an integral Hilbert-Schmidt operator  $A$  is in fact completely continuous and moreover an operator of the Schmidt class, with

$$\|A\|_2^2 = \int_a^b \int_a^b |K^2(s, t)| ds dt.$$

Let  $\{f_k(s)\}$  and  $\{\varphi_i(t)\}$  be complete orthonormal systems in  $L^2[a, b]$  with arguments  $s$  and  $t$  respectively. Then the system  $\{f_k(s)\varphi_i(t)\}$  is a complete orthonormal system in  $L^2([a, b] \times [a, b])$ . Indeed, the orthonormality can be verified immediately. We shall prove completeness. Let the function  $\omega(s, t)$  belong to  $L^2([a, b] \times [a, b])$  and be orthogonal to each function of the system  $\theta_{k,i} = \{f_k(s)\varphi_i(t)\}$ , i.e.,  $(\omega, \theta_{k,i}) = 0$  for all  $k$  and  $i$ . Then, as we know, the function

$$\omega_k(t) = \int_a^b \omega(s, t) f_k(s) ds$$



belongs to  $L^2[a, b]$ . Therefore

$$(\omega, \theta_{k,i}) = (\omega, f_k \varphi_i) = (\omega_k, \varphi_i).$$

Thus the relation  $(\omega, \theta_{k,i}) = 0$  for all  $k$  and  $i$  implies that  $(\omega_k, \varphi_i) = 0$  for all  $k$  and  $i$ , and by the completeness of the system  $\{\varphi_i\}$  it follows that  $\omega_k = 0$  for all  $k$ . Now  $\omega_k(t)$  is defined for almost all  $t$  and for each  $k$  it is equal to zero for almost every  $t$ . Since  $\{f_k\}$  is complete, it follows that  $\omega(s, t) = 0$  for almost every  $s$  and  $t$ , i.e.,  $\omega = 0$  as an element of  $L^2([a, b] \times [a, b])$  and the system  $\{f_k(s)\varphi_i(t)\}$  is complete.

We now return to the calculation of the absolute norm of the operator  $A$ . By the completeness and orthonormality of the system  $\{f_k \varphi_i\}$  we have

$$\begin{aligned} \|A\|_2^2 &= \sum_{k,i=1}^{\infty} |(Af_k, \varphi_i)|^2 = \sum_{k,i=1}^{\infty} \left| \int_a^b \int_a^b K(s, t) f_k(s) \varphi_i(t) ds dt \right|^2 \\ &= \int_a^b \int_a^b |K(s, t)|^2 ds dt. \end{aligned}$$

Thus under the hypothesis that  $\int_a^b \int_a^b |K(s, t)|^2 ds dt < \infty$ , the operator

$A\varphi = \int_a^b K(s, t)\varphi(t) dt$  is an operator of the Schmidt class.

#### 2.4. The Fredholm Alternative

It is often necessary to find solutions of various integral or differential equations.

An example of an integral equation is the equation

$$\varphi(s) = \int_a^b K(s, t)\varphi(t) dt + f(s),$$

where  $f$  and  $K$  are given functions and  $\varphi$  is the unknown function. Mathematicians have considered and studied particular kinds of integral equations for a long time. Thus, for example, as early as 1823 Abel considered an equation of a form slightly different from the one given above, namely the equation

$$f(s) = \int_0^s \frac{\varphi(t)}{(s-t)^\alpha} dt, \quad 0 < \alpha < 1, \quad f(0) = 0,$$

where  $f$  is a given function and  $\varphi$  the unknown, and showed that the solution of this equation has the form

$$\varphi(t) = \frac{\sin \pi \alpha}{\pi} \int_0^t \frac{f'(s)}{(t-s)^{1-\alpha}} ds.$$

Many differential equations reduce to integral equations.

For example consider the well-known Sturm-Liouville equation

$$-y'' + q(x)y = \rho^2 y,$$

where  $q(x)$  is a known function,  $\rho$  is some numerical parameter, and  $y$  is the function to be determined.

By the method of variation of parameters, for example, it is not difficult to verify that the solution of the Cauchy problem ( $y(0) = 0$ ,  $y'(0) = 1$ ) of a given equation can be found from the following integral equation:

$$y(x) = \frac{1}{\rho} \sin \rho x + \frac{1}{\rho} \int_0^x \sin \rho(x - \xi) q(\xi) y(\xi) d\xi.$$

In this situation it is important to emphasize that the integral equations under consideration, which in many cases can be written in the form

$$\varphi = K\varphi + f,$$

( $\varphi$  is the unknown function,  $K$  is the integral operator, and  $f$  a given function), frequently turn out to be equations with a completely continuous operator  $K$ .

We shall now prove an important theorem regarding the solvability of such equations, and the specific form of the operator  $K$  will not be of interest to us in what follows. Throughout the following we shall assume that  $K$  is an arbitrary completely continuous operator defined on a Hilbert space  $H$ .

Setting  $A = E - K$ , where  $E$  is the identity operator, we can rewrite the equation in the form

$$A\varphi = f.$$

Along with this equation we consider in  $H$  the homogeneous equation

$$A\varphi_0 = 0$$

and two equations with the adjoint operator—one inhomogeneous

$$A^*\varphi = g$$

and one homogeneous

$$A^*\varphi_0 = 0.$$

**THEOREM** (The Fredholm alternative). *The following assertions are true:*



a) The inhomogeneous equation  $A\varphi = f$  is solvable if and only if  $f \in H$  is orthogonal to all solutions of the adjoint equation  $A^*\varphi_0 = 0$ .

b) Either the homogeneous equation  $A\varphi_0 = 0$  has a nonzero solution, or the inhomogeneous equation has a unique solution for any  $f \in H$ .

c) The homogeneous equations  $A\varphi_0 = 0$  and  $A^*\varphi_0 = 0$  have the same finite number of linearly independent solutions.

We make a number of observations in connection with the theorem just stated.

First of all, if the operator  $K$  were, for example, an integral operator with a degenerate kernel of the form  $K(s, t) = \sum_{i=1}^n P_i(s)Q_i(t)$ , it would be easy to verify that all the equations reduce to systems of linear algebraic equations, for which all the assertions of the theorem reduce to propositions known from linear algebra. An integral operator with an arbitrary kernel  $K(s, t) \in L^2([a, b] \times [a, b])$  can be approximated by an operator with a degenerate kernel, thereby proving the theorem. We emphasize that we are considering abstract operators in  $H$ , and consequently we shall give a proof not connected with the study of degenerate kernels.

Second, all that has been said above can be carried over exactly to the case when the operator  $A$  maps a Banach space  $B_1$  into a Banach space  $B_2$ , and a variety of new and widely used concepts arises in this theory, which we shall now explain. Let  $\ker(A) = Z(A)$  be the *kernel* of the operator  $A$ , i.e., the set of solutions of the equation

$$A\varphi_0 = 0, \quad \varphi_0 \in B_1.$$

Let  $R(A)$  denote the *image* of the operator  $A$  in  $B_2$ , i.e., the set of  $f \in B_2$  for which the equation  $A\varphi = f$  is solvable. It is clear that  $\ker A$  is a subspace (a closed linear manifold), being the preimage of 0 under a continuous mapping. The set  $R(A)$  is not always closed. Analogously one can define  $\ker A^* = Z(A^*)$  and  $R(A^*)$ . If  $R(A)$  and  $R(A^*)$  are subspaces, one can define new Banach spaces (quotient spaces)

$$\text{coker } A = B_2/R(A) \quad \text{and} \quad \text{coker } A^* = B_1^*/R(A^*).$$

They are called the *cokernels* of the operators  $A$  and  $A^*$  respectively.

Let

$$\alpha(A) = \dim \ker A, \quad \beta(A) = \dim \text{coker } A, \quad i(A) = \alpha(A) - \beta(A),$$

where  $\dim$  denotes dimension. The operator  $A$  is called a *Fredholm operator* if the numbers  $\alpha(A)$  and  $\beta(A)$  are finite. In this case  $i(A)$  is called the *index* of the operator  $A$ .

In the finite-dimensional case, where  $\dim B_1 = N_1$  and  $\dim B_2 = N_2$ , it is easy to verify the equalities

$$\begin{aligned} N_1 - \alpha(A) &= N_2 - \beta(A) = \text{rank } A, \\ N_2 - \alpha(A^*) &= N_1 - \beta(A^*) = \text{rank } A^*, \end{aligned}$$

and since  $\text{rank } A = \text{rank } A^*$ , (the theorem that the row rank of a matrix equals its column rank), we have

$$\alpha(A) = \beta(A^*), \quad \beta(A) = \alpha(A^*), \quad i(A) = -i(A^*).$$

Having made these general remarks, we now turn directly to the proof of the Fredholm alternative in the case when the operators  $A$  and  $A^*$  map a Hilbert space  $H$  into itself.

PROOF: We first show that in the case under consideration the linear manifold  $R(A)$ —the range of values of the operator  $A$ —is closed, that the linear manifold  $R(A^*)$  is closed, and that the following decompositions hold:

$$\begin{aligned} H &= Z(A) \oplus R(A^*), \\ H &= Z(A^*) \oplus R(A). \end{aligned}$$

Indeed, let  $f_n \in R(A)$  and  $f_n \rightarrow f$ . There exist vectors  $\varphi_n \in H$  such that

$$f_n = A\varphi_n = \varphi_n - K\varphi_n,$$

where  $A = E - K$ . We can always assume that the vectors  $\varphi_n$  are orthogonal to  $Z(A)$  (by subtracting from  $\varphi_n$  its projection on  $Z(A)$  if necessary). We shall show that  $\|\varphi_n\|$  are uniformly bounded. In fact, if this is not the case, then there exists a subsequence for which  $\|\varphi_{n_k}\| \rightarrow \infty$ . Therefore  $\frac{\varphi_{n_k}}{\|\varphi_{n_k}\|} - K \frac{\varphi_{n_k}}{\|\varphi_{n_k}\|} \rightarrow 0$  as  $n_k \rightarrow \infty$ . Again passing to a subsequence  $\{\tilde{\varphi}_{n_k}\}$ , we can assume by the complete continuity of  $K$  that the sequence  $\left\{ \frac{K\tilde{\varphi}_{n_k}}{\|\tilde{\varphi}_{n_k}\|} \right\}$

converges. Therefore  $\left\{ \frac{\tilde{\varphi}_{n_k}}{\|\tilde{\varphi}_{n_k}\|} \right\}$  will converge, say, to the vector  $z \in H$ . Then  $\|z\| = 1$  and  $Az = 0$ , i.e.,  $z \in Z(A)$ . However, we are assuming the vectors  $\varphi_n$  to be orthogonal to  $Z(A)$ , and therefore  $z \perp Z(A)$  also. We have now reached a contradiction. Consequently the set  $\|\varphi_n\|$  is uniformly bounded. In addition in this case the sequence  $\{K\varphi_n\}$  can be assumed convergent, and then it follows from the equation  $f_n = \varphi_n - K\varphi_n$  that



$\{\varphi_n\}$  converges. Let  $\varphi = \lim_{n \rightarrow \infty} \varphi_n$ . Then  $f = A\varphi$ , which was required. For  $R(A^*)$  the proof is similar.

We now prove the validity of the decomposition for  $H$ . Let  $h \in Z(A)$ . Then  $(h, A^*\varphi) = (Ah, \varphi) = 0$  for all  $\varphi \in H$ , i.e.,  $Z(A) \perp R(A^*)$ . It remains only to remark that no nonzero vector  $z$  can be simultaneously orthogonal to  $Z(A)$  and  $R(A^*)$ , i.e., that in fact  $H = Z(A) \oplus R(A^*)$ . Suppose there exists  $z \in H$  and such that  $z \perp R(A^*)$ . Then for any  $\varphi \in H$  we have  $(Az, \varphi) = (z, A^*\varphi) = 0$ , i.e.,  $z \in Z(A)$ , which was to be proved.

Assertion a) of the theorem follows from the assertions that have now been proved. In fact  $f \perp Z(A^*)$  if and only if  $f \in R(A)$ , i.e., there exists a vector  $\varphi$  such that  $A\varphi = f$ .

We now turn to the proof of assertion b) of the theorem.

For each integer  $k$  we set  $H^k = R(A^k)$  (in particular,  $H^1 = R(A)$ ). The inclusions  $H \supset H^1 \supset H^2 \supset \dots$  are obvious, and by what has been proved all  $H^k$  are closed. Moreover  $A(H^k) = H^{k+1}$ .

We shall show first that there exists an index  $l$  such that  $H^{k+1} = H^k$  for all  $k \geq l$ , and also that if  $Z(A) = \{0\}$ , then  $R(A) = H$ , and conversely, if  $R(A) = H$ , then  $Z(A) = \{0\}$ .

In fact if such an  $l$  does not exist, then all  $H^k$  are distinct. We construct an orthonormal sequence  $\{\varphi_k\}$  such that  $\varphi_k \in H^k$  and  $\varphi_k \perp H^{k+1}$ . Let  $p > k$ ; then

$$K\varphi_p - K\varphi_k = -\varphi_k + (\varphi_p + A\varphi_k - A\varphi_p)$$

and consequently  $\|K\varphi_p - K\varphi_k\| \geq 1$  since  $\varphi_p + A\varphi_k - A\varphi_p \in H^{k+1}$ . Therefore no convergent subsequence can be chosen from the sequence  $\{K\varphi_k\}$ , contradicting the complete continuity of the operator  $K$ .

Furthermore, if  $Z(A) = \{0\}$ , then the operator  $A$  is one-to-one and consequently if  $R(A) \neq H$ , then the chain  $\{H^k\}$  consists of distinct spaces, contradicting what was said above. Therefore  $R(A) = H$  and analogously  $R(A^*) = H$  if  $Z(A^*) = \{0\}$ .

But if  $R(A) = H$ , then it follows from the decomposition  $H = R(A) \oplus Z(A^*)$  that  $Z(A^*) = \{0\}$ , and then  $R(A^*) = H$ . From the decomposition  $H = R(A^*) \oplus Z(A)$ , we find that  $Z(A) = \{0\}$ , which was to be proved.

Thus assertion b) of the theorem is proved. Indeed, the assertions proved above, that if  $Z(A) = \{0\}$ , then  $R(A) = H$  and if  $R(A) = H$ , then  $Z(A) = \{0\}$ , constitute part b) of the theorem.

We prove finally part c) of the theorem. Assume that the space  $Z(A)$  is infinite-dimensional. Then in it there exists an infinite orthonormal system  $\{\varphi_k\}$  and  $K\varphi_k = \varphi_k$ . Consequently if  $n \neq m$ , then  $\|K\varphi_n - K\varphi_m\| = \sqrt{2}$ , and no convergent subsequence can be chosen from the sequence  $\{K\varphi_k\}$ , contradicting the complete continuity of the operator  $K$ .



Let  $\alpha = \dim Z(A)$  be the dimension of the space  $Z(A)$ ,  $\beta = \dim Z(A^*)$  the dimension of  $Z(A^*)$ . Assume that  $\dim Z(A) < \dim Z(A^*)$ . Choose an orthonormal basis  $\{\varphi_1, \dots, \varphi_\alpha\}$  in  $Z(A)$  and let  $\{\psi_1, \dots, \psi_\beta\}$  be an orthonormal basis of  $Z(A^*)$ . Let  $T\varphi = A\varphi + \sum_{j=1}^{\alpha} (\varphi, \varphi_j)\psi_j$ . Since the operator  $T$  is obtained from the operator  $A$  by adding a finite-dimensional operator, all the results proved above for  $A$  remain true for  $T$  also. We shall show that the equation  $T\varphi = 0$  has only the trivial solution. Let  $A\varphi + \sum_{j=1}^{\alpha} (\varphi, \varphi_j)\psi_j = 0$ . Since all the vectors  $\psi_j$  are orthogonal to vectors of the form  $A\varphi$ , we find that  $A\varphi = 0$  and  $(\varphi, \varphi_j) = 0$  for  $1 \leq j \leq \alpha$ .

Therefore the vector  $\varphi$  is on the one hand a linear combination of the vectors  $\varphi_j$  and on the other hand orthogonal to them; it follows that  $\varphi = 0$ .

Thus the equation  $T\varphi = 0$  has only the trivial solution, but then according to part b) of the theorem there exists a vector  $f$  such that  $Af + \sum_{j=1}^{\alpha} (f, \varphi_j)\psi_j = \psi_{\alpha+1}$ .

Taking the inner product of this equation with  $\psi_{\alpha+1}$ , we obtain 0 on the left and 1 on the right. Thus the assumption that  $\alpha < \beta$  is false. Consequently  $\alpha \geq \beta$ , i.e.,  $\dim Z(A) \geq \dim Z(A^*)$ . Interchanging the operators  $A$  and  $A^*$ , we find that  $\beta \geq \alpha$ , and consequently  $\dim Z(A) = \dim Z(A^*)$ .

The theorem (the Fredholm alternative) is now proved completely. ■

## 2.5. Projections

Let  $G$  be some subspace of a Hilbert space  $H$  and  $F$  its orthogonal complement, i.e.,  $H = G \oplus F$  or  $F = H \ominus G$ . This means that each vector  $h \in H$  is uniquely representable in the form  $h = g + f$ ,  $g \in G$ ,  $f \in F$ . The vector  $g$  is called the *projection* of  $h$  on  $G$ .

The operator defined on all of a Hilbert space  $H$  assigning to each vector  $h \in H$  its projection on the subspace  $G$  is called the *projection* on  $G$  (or the operator of projection on  $G$  or the orthoprojection on  $G$ ) and is denoted by the symbol  $P$  or  $P_G$ , so that

$$g = Ph = P_G h.$$

A projection is obviously linear. In addition it is bounded and its norm is 1. Indeed, since  $\|h\|^2 = \|g\|^2 + \|f\|^2$ , we have  $\|g\| \leq \|h\|$ , i.e.,  $\|Ph\| \leq \|h\|$ . However, if  $h \in G$ , then  $g = h$ , so that  $\|Ph\| = \|h\|$ . Consequently  $\|P\| = 1$ .

For projections we have the equality  $P^2 = P$ . Indeed, for any vector  $h \in H$  the vector  $g = Ph \in G$  and therefore  $P^2h = Ph$ , i.e.,  $P^2 = P$ . Let  $h_1, h_2 \in H$ ,  $h_1 = g_1 + f_1$ ,  $h_2 = g_2 + f_2$ . In such a case

$$(g_1, h_2) = (g_1, g_2) = (h_1, g_2), \quad (Ph_1, h_2) = (h_1, Ph_2),$$

i.e.,  $P^* = P$ . A projection operator is determined by these two properties.

**THEOREM 4.** *If  $P$  is a linear operator defined everywhere on  $H$  and for any  $h_1, h_2 \in H$*

$$(P^2 h_1, h_2) = (Ph_1, h_2), \quad (Ph_1, h_2) = (h_1, Ph_2),$$

*then there exists a subspace  $G \subset H$  on which the projection operator is  $P$ .*

**PROOF:** The operator  $P$  is bounded:

$$\|Ph\|^2 = (Ph, Ph) = (P^2 h, h) = (Ph, h), \quad \|Ph\|^2 \leq \|Ph\| \cdot \|h\|,$$

i.e.,  $\|Ph\| \leq \|h\|$ .

We denote by  $G$  the set of vectors  $g \in H$  for which  $Pg = g$ . It is clear that  $G$  is a linear manifold; we shall prove that it is closed, hence is a subspace. Let  $g_n \in G$ ,  $n = 1, 2, \dots$  and  $g_n \rightarrow g$ . Then  $Pg_n = g_n$  and  $Pg - g_n = Pg - Pg_n = P(g - g_n)$ , from which  $\|Pg - g_n\| \leq \|g - g_n\|$ . Letting  $n$  tend to infinity, we obtain  $\|Pg - g\| \leq 0$ , i.e.,  $Pg = g$ , and it follows that  $G$  is closed.

We denote the projection on  $G$  by  $P_G$ . We shall show that  $P_G = P$ . For any  $h \in H$  the vector  $Ph = g$  belongs to  $G$ , since  $P(Ph) = Ph$ . The vector  $P_G h$  also belongs to the subspace  $G$ . Therefore we need to prove that  $(Ph - P_G h, g') = 0$ , or  $(Ph, g') = (P_G h, g')$  for any  $g' \in G$ . But this follows from the fact that

$$\begin{aligned} (Ph, g') &= (h, Pg') = (h, g'), \\ (P_G h, g') &= (h, P_G g') = (h, g'). \blacksquare \end{aligned}$$

We remark that if  $P$  is the projection on  $G$ , then  $E - P$ , where  $E$  is the identity operator, is also a projection, and  $E - P$  projects onto  $H \ominus G$ . Indeed  $(E - P)^* = E - P$ ,  $(E - P)^2 = E - P$ , and for any vector  $h \in H$  we have  $(E - P)h = h - g = f$ , where  $f \in H \ominus G$ .

**THEOREM 5.** *The composition of two projections  $P_{G_1}$  and  $P_{G_2}$  is a projection if and only if the two operators commute:*

$$P_{G_1} P_{G_2} = P_{G_2} P_{G_1};$$

*if this condition is met, then  $P_{G_1} P_{G_2} = P_G$ , where  $G = G_1 \cap G_2$ .*

**PROOF:** If the composition of the two operators  $P_{G_1} P_{G_2}$  is a projection, then

$$P_{G_1} P_{G_2} = (P_{G_1} P_{G_2})^* = P_{G_2}^* P_{G_1}^* = P_{G_2} P_{G_1}.$$



The vector  $g = P_{G_1} P_{G_2} h = P_{G_2} P_{G_1} h$  belongs to  $G_1$  by the first representation, and to  $G_2$  by the second, i.e., it belongs to  $G_1 \cap G_2$ , and consequently  $G \subset G_1 \cap G_2$ . Since the opposite inclusion is obvious, the necessity is proved.

Now suppose  $P_{G_1} P_{G_2} = P_{G_2} P_{G_1} = P$ . It follows from this that  $P^2 = (P_{G_1} P_{G_2})^2 = P_{G_1} P_{G_2} P_{G_1} P_{G_2} = P_{G_1} P_{G_1} P_{G_2} P_{G_2} = P_{G_1} P_{G_2} = P$  and  $P^* = (P_{G_1} P_{G_2})^* = P_{G_2}^* P_{G_1}^* = P_{G_2} P_{G_1} = P$ . But these two properties show that  $P_{G_1} P_{G_2}$  is a projection. ■

**COROLLARY.** Two subspaces  $G_1$  and  $G_2$  are orthogonal if and only if  $P_{G_1} P_{G_2} = 0$ .

We shall state below three propositions which we shall make use of and whose proofs will be given later.

**PROPOSITION 3.** *The sum of projections*

$$P_{G_1} + P_{G_2} + \cdots + P_{G_n} = Q \quad (n < \infty)$$

is a projection if and only if  $P_{G_j} P_{G_k} = 0$  ( $j \neq k$ ), i.e., if and only if the subspaces  $G_i$  ( $i = 1, 2, \dots, n$ ) are pairwise orthogonal, and in this case  $Q = P_G$ , where  $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$ .

**PROPOSITION 4.** *The difference of two projections  $P_{G_1} - P_{G_2}$  is a projection if and only if  $G_2 \subset G_1$ , and in that case  $P_{G_1} - P_{G_2}$  is the projection on  $G_1 \ominus G_2$ .*

**PROPOSITION 5.** *The relation  $G_2 \subset G_1$  is equivalent to the inequality  $\|P_{G_2} f\| \leq \|P_{G_1} f\|$  or the inequality  $(P_{G_2} f, f) \leq (P_{G_1} f, f)$  for any  $f \in H$ .*

**PROOF OF PROPOSITION 3.** **NECESSITY:** Let the operator  $Q$  be a projection. For two distinct indices  $j$  and  $k$  the inequality  $0 \leq (Pf, f) \leq \|f\|^2$ , which holds for any  $f$  and any projection  $P$ , implies

$$(P_{G_j} f, f) + (P_{G_k} f, f) \leq \sum_{i=1}^n (P_{G_i} f, f) = (Qf, f) \leq \|f\|^2.$$

Setting  $f = P_{G_j} h$ , we obtain

$$\|P_{G_j} h\|^2 + (P_{G_k} P_{G_j} h, P_{G_j} h) \leq \|P_{G_j} h\|^2.$$

Consequently

$$(P_{G_k} P_{G_j} h, P_{G_j} h) = \|P_{G_k} P_{G_j} h\|^2 = 0.$$

Since the last equality holds for any vector  $h$ , we have

$$P_{G_k} P_{G_j} = 0$$



and by the corollary to Theorem 5 we obtain  $G_j \perp G_k$ .

The sufficiency of the condition follows from Theorem 4. By the relation  $P_{G_j} P_{G_k} = 0$ ,  $j \neq k$  we have  $Q^2 = Q$  and  $Q^* = Q$ . The equality  $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$  is obvious. ■

**PROOF OF PROPOSITION 4. SUFFICIENCY:** Suppose  $G_2 \subset G_1$ . Consider the subspace  $G = G_1 \ominus G_2$ . Since  $G \perp G_1$  and  $G \oplus G_2 = G_1$ , Proposition 3 implies that the operator  $P_{G_1} = P_G + P_{G_2}$  is a projection. From this we find that the difference  $P_{G_1} - P_{G_2}$  is a projection.

**NECESSITY:** Let the operator  $P_G = P_{G_1} - P_{G_2}$  be a projection. We find that the sum

$$P_G + P_{G_2} = P_{G_1}$$

is a projection and so from Proposition 3 we have

$$G \oplus G_2 = G_1, \quad \text{i.e., } G_2 \subset G_1. \quad \blacksquare$$

**PROOF OF PROPOSITION 5. NECESSITY:** Let  $G_2 \subset G_1$ . By Proposition 4 we have a projection

$$P_G = P_{G_1} - P_{G_2}, \quad \text{where } G = G_1 \ominus G_2.$$

Therefore for any  $f \in H$

$$(P_G f, f) = (P_{G_1} f, f) - (P_{G_2} f, f) \geq 0.$$

**SUFFICIENCY:** Consider the subspaces  $V_1 = H \ominus G_1$  and  $V_2 = H \ominus G_2$ .

Let  $f \in V_1$ . Then  $P_{G_1} f = 0$ . Hence by the inequality  $\|P_{G_2} f\| \leq \|P_{G_1} f\|$  for any vector  $f \in H$  we have  $P_{G_2} f = 0$ , i.e.,  $f \in V_2$ . Therefore  $V_1 \subset V_2$ , which is equivalent to the inclusion  $G_2 \subset G_1$ . ■

## 2.6. The Spectrum of an Operator

One of the important problems connected with the study of linear operators in a Hilbert or Banach space is that of finding elements whose direction is preserved under the action of the operator, i.e., elements satisfying the equation  $Af = \lambda f$ , where  $\lambda$  is a number. Each such element  $f \neq 0$  is called an *eigenvector* of the operator and  $\lambda$  is called an *eigenvalue*.

The solutions of the equation  $Af = \lambda_1 f$ , where  $\lambda_1$  is fixed, obviously form a subspace  $H_{\lambda_1}$  in view of the linearity and continuity of the operator  $A$ . The dimension of this subspace is called the *multiplicity* of the eigenvalue  $\lambda_1$  and the subspace is called the *eigenspace* corresponding to the eigenvalue  $\lambda_1$ . If  $\dim H_{\lambda_1} = 1$ , then the eigenvalue  $\lambda_1$  is called *simple*.

The definitions given above for eigenvalues and eigenvectors generalize concepts known from linear algebra. In particular the set of all eigenvalues is called the *spectrum* of a matrix in linear algebra.

In the general case of a bounded linear operator on infinite-dimensional space the situation is more complicated. We give the definition of the spectrum of an operator.

Consider the operator  $A - \lambda E = B(\lambda)$ . Suppose that for some  $\lambda$  the operator  $A - \lambda E$  has an inverse  $R_\lambda = (A - \lambda E)^{-1}$ . The operator  $R_\lambda$  is called the *resolvent* of the operator  $A$ . The values of  $\lambda$  for which  $R_\lambda$  exists, is defined on the entire space, and is bounded are called the *regular* values of the operator  $A$  (or are said to belong to the *resolvent set* of the operator). The set of all values of  $\lambda$  that are not regular is called the *spectrum* of the operator  $A$ ; in particular all the eigenvalues belong to the spectrum.

Thus the spectrum of an operator is the complement (in the complex plane) of the resolvent set.

The following assertions are consequences of the considerations of Chapter 2 on inverse operators:

a) If  $\lambda$  is such that  $\frac{1}{|\lambda|} \|A\| = q < 1$ , then the operator  $A - \lambda E$  has a bounded inverse, and  $R_\lambda = -\frac{1}{\lambda} \left( + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \dots \right)$ . Thus the spectrum of the operator  $A$  is contained in the set  $|\lambda| \leq \|A\|$ . This result will be sharpened in Sec. 4.3.

b) If  $\lambda$  is a regular value, then  $\lambda + \Delta\lambda$  is also a regular value for  $|\Delta\lambda| < \|(A - \lambda E)^{-1}\|^{-1}$ . It follows from this that the set of regular values (the resolvent set) is an open set and the spectrum, being its complement, is a closed set.

In a finite-dimensional space, as we know, there are only two possibilities:

—The equation  $Af = \lambda f$  has a nonzero solution, i.e.,  $\lambda$  is an eigenvalue of the operator (matrix)  $A$ ; the operator  $(A - \lambda E)^{-1}$  does not exist in this case;

—There exists a bounded operator  $(A - \lambda E)^{-1}$  (defined on the entire space), i.e.,  $\lambda$  is a regular value.

In an infinite-dimensional space there is a third possibility:

—The operator  $(A - \lambda E)^{-1}$  exists, i.e., the equation  $Af = \lambda f$  has only the zero solution, but this operator is not defined on the entire space.\*

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\*Thus the values belonging to the spectrum are those and only those at which  $A - \lambda E$  fails to have a bounded inverse defined on the entire space.



In this connection the spectrum of an operator is divided into three parts:

1. The *point spectrum*—the values of  $\lambda$  for which there exists a nonzero solution of the equation  $Af = \lambda f$ . The point spectrum of an operator obviously coincides with the set of eigenvalues of the operator.

2. The *continuous spectrum*—the values of  $\lambda$  for which the operator  $B(\lambda) = (A - \lambda E)$  has an inverse  $B^{-1}(\lambda) = R_\lambda$  with a dense domain of definition, but this domain does not coincide with the entire space.

3. The *residual spectrum*—the values of  $\lambda$  for which the operator  $B(\lambda) = (A - \lambda E)$  has an inverse  $B^{-1}(\lambda) = R_\lambda$ , but its domain of definition is not dense in the entire space.

Let us consider some examples.

#### EXAMPLES

1. In the Banach space  $C[0, 1]$  consider the operation of multiplication by the independent variable:

$$Af(x) = xf(x).$$

We shall show that every real number  $\lambda \in [0, 1]$  belongs to the residual spectrum of this operator. This operator has no eigenvalues and no continuous spectrum, since if  $Af(x) = xf(x) = \lambda f(x)$ , then  $(x - \lambda)f(x) = 0$  on the closed interval  $[0, 1]$ . Therefore  $f(x) \equiv 0$ . Thus the homogeneous equation has only the trivial solution, and the operator  $R_\lambda = B^{-1}(\lambda) = (A - \lambda E)^{-1}$  exists. The domain of definition of  $R_\lambda$  consists of functions that must vanish at the point  $\lambda \in [0, 1]$ . Consequently the domain of definition of  $R_\lambda$  is not dense in  $C[0, 1]$ , i.e., every number  $\lambda$  belonging to the closed interval  $[0, 1]$  is a point of the residual spectrum of the operator.

2. Let the operator  $A$  be the shift operator defined in the Hilbert space  $l^2$ : if  $\xi = \{\xi_n\}_{n=1}^\infty \in l^2$ , then

$$A\xi = A(\xi_1, \xi_2, \dots) = (0, \xi_1, \xi_2, \dots).$$

The operator  $A^{-1} = (A - 0 \cdot E)^{-1}$  exists and establishes a one-to-one mapping, but is defined only on sequences for which the first coordinate equals zero. The set of such sequences is not dense in  $l^2$ . Consequently the point  $\lambda = 0$  belongs to the residual spectrum of the operator  $A$ .

There are various reasons why this may happen. Different classifications of the spectrum are connected with the reasons for the failure.



Thus we have introduced the following classification of the spectrum of a bounded linear operator: it consists of three pairwise disjoint sets known as the point spectrum, the continuous spectrum, and the residual spectrum.

### 2.7. Symmetric Operators. Properties of the Quadratic Form of an Operator

We now take up the study of bounded symmetric operators defined on all of a Hilbert space.

The quadratic form of a symmetric operator plays an important role in studying the operator.

In the case of a symmetric operator  $A$  the values of the *quadratic form*  $(Af, f)$  corresponding to it are always real, since

$$(Af, f) = (f, A^* f) = (f, Af) = \overline{(Af, f)}.$$

The converse assertion holds also: if for some operator  $A$  on a complex Hilbert space the quadratic form  $(Af, f)$  is real-valued, then  $A$  is a symmetric operator. Indeed we always have the equality

$$\begin{aligned} (A(f+g), f+g) - (A(f-g), f-g) \\ + i(A(f+ig), f+ig) - i(A(f-ig), f-ig) = 4(Af, g). \end{aligned}$$

Interchanging  $f$  and  $g$  in this relation and taking complex conjugates, we obtain

$$\begin{aligned} (f+g, A(f+g)) - (f-g, A(f-g)) \\ + i(f+ig, A(f+ig)) - i(f-ig, A(f-ig)) = 4(f, Ag). \end{aligned}$$

If the quadratic form  $(Af, f)$  assumes only real values, then  $(f, Af) = \overline{(Af, f)} = (Af, f)$ , and the left-hand sides of these two equalities coincide. It follows from this that  $(Af, g) = (f, Ag)$ , i.e.,  $A^* = A$ .

We remark that this proposition holds only in complex Hilbert space.

If the quadratic form is real-valued, then all the eigenvalues of the operator are also real. In this case an eigenvalue  $\lambda$  has the form  $\lambda = (Af, f)/(f, f)$ , where  $f$  is an eigenvector.

The eigenvectors  $f$  and  $g$  corresponding to different eigenvalues  $\lambda$  and  $\mu$  are orthogonal. Indeed,  $\lambda(f, g) = (\lambda f, g) = (Af, g) = (f, Ag) = (f, \mu g) = \mu(f, g)$ , so that  $(f, g) = 0$ , if  $\lambda \neq \mu$ .

For the quadratic form  $(Af, f)$  we have the inequalities

$$|(Af, f)| \leq \|Af\| \|f\| \leq \|A\| \|f\|^2.$$

Let the smallest constant  $M$  for which the inequality

$$|(Af, f)| \leq M \|f\|^2$$

holds for all  $f$  be denoted by  $N_A$ . Then  $N_A \leq \|A\|$ .

This inequality is also true for a nonsymmetric operator. But if the operator is symmetric, then these two constants are equal. In fact since  $(A^2 f, f) = (Af, Af)$ , we have for any  $\lambda > 0$

$$\begin{aligned} \|Af\|^2 &= \frac{1}{4} \left[ \left( A \left( \lambda f + \frac{1}{\lambda} Af \right), \lambda f + \frac{1}{\lambda} Af \right) \right. \\ &\quad \left. - \left( A \left( \lambda f - \frac{1}{\lambda} Af \right), \lambda f - \frac{1}{\lambda} Af \right) \right] \\ &\leq \frac{1}{4} \left[ N_A \left\| \lambda f + \frac{1}{\lambda} Af \right\|^2 + N_A \left\| \lambda f - \frac{1}{\lambda} Af \right\|^2 \right] \\ &= \frac{1}{2} N_A \left[ \lambda^2 \|f\|^2 + \frac{1}{\lambda^2} \|Af\|^2 \right]. \end{aligned}$$

If  $\|Af\| \neq 0$ , then the last expression attains its minimum value for  $\lambda^2 = \|Af\|/\|f\|$ , from which it follows that

$$\|Af\|^2 \leq N_A \|Af\| \cdot \|f\|, \quad \|Af\| \leq N_A \|f\|,$$

and these inequalities hold also when  $\|Af\| = 0$ . Thus  $\|A\| \leq N_A$ . Consequently  $\|A\| = N_A$ . Thus we have proved the following theorem.

**THEOREM 6.** *If  $A$  is a bounded symmetric linear operator defined on the whole space  $H$ , then all its eigenvalues are real, the eigenvectors corresponding to distinct eigenvalues are orthogonal, the quadratic form  $(Af, f)$  is real-valued, and the smallest constant  $N_A$  for which  $|(Af, f)| \leq N_A \|f\|^2$  is equal to  $\|A\|$ .*

We remark further that the sum of symmetric operators is a symmetric operator, as is a linear combination of them with real coefficients. By the continuity of the inner product, limits in norm, pointwise limits, and weak limits of symmetric operators are symmetric operators. The composition of symmetric operators is a symmetric operator if and only if the factors commute, i.e.,  $A_1 A_2 = A_2 A_1$  (we write  $A_1 \sim A_2$ ).

There is an order relation on symmetric operators defined by specifying that  $A \geq B$  if  $(Af, f) \geq (Bf, f)$  for all  $f \in H$ .

A linear operator with the property  $(Af, f) \geq 0$  is called *positive*. A positive operator in a complex Hilbert space is symmetric. This follows from



the fact that in this case the quadratic form is real-valued, and it was proved above that in such a case  $A$  is symmetric.

For a positive symmetric operator we have a *generalized Schwarz inequality*:  $|(Af, g)|^2 \leq (Af, f)(Ag, g)$ . Indeed, let  $h_\lambda = f + \lambda(Af, g)g$  and let  $\lambda$  be any real number. We then have  $0 \leq (Ah_\lambda, h_\lambda) = (Af, f) + 2\lambda|(Af, g)|^2 + \lambda^2|(Af, g)|^2(Ag, g)$  and the discriminant of this quadratic trinomial must be nonpositive. The proof follows from this.

The *lower and upper bounds* of a symmetric operator  $A$  are the respectively the largest  $m$  and the smallest  $M$  among the numbers for which the inequalities

$$m(f, f) \leq (Af, f) \leq M(f, f)$$

hold, i.e., the inequalities  $mE \leq A \leq ME$ . In other words  $M$  is the supremum and  $m$  the infimum of the values of the quadratic form  $(Af, f)$  when  $f$  is subject to the condition  $\|f\| = 1$ . But, as we have shown in the preceding theorem, the upper bound of  $|(Af, f)|$  is the norm of the operator  $A$ , and consequently

$$\|A\| = \max\{|m|, |M|\}.$$

From this it follows in particular that the relations  $A \geq B$  and  $A \leq B$  can hold only when  $A = B$ . In fact if they both hold, then for the operator  $C = A - B$  we shall have  $(Cf, f) = 0$ , so that  $m_C = M_C = 0$  and therefore  $\|C\| = 0$ .

The order relation just introduced between operators is obviously transitive, i.e., if  $A \geq B$  and  $B \geq C$ , then  $A \geq C$ . Moreover, if  $A \geq B$ , then  $A + C \geq B + C$  and  $kA \geq kB$  for any operator  $C = C^*$  and any number  $k > 0$ .

Although these properties of the order relation resemble the properties of real numbers, there is an important difference between the two: it is possible to exhibit two symmetric operators, neither of which is larger than the other in this sense. We see that the set of symmetric operators is only partially ordered.

**THEOREM 7.** *Every bounded monotonic sequence of symmetric operators  $\{A_n\}$  converges pointwise to some symmetric operator.*

**PROOF:** Without loss of generality we shall suppose that

$$0 \leq A_1 \leq A_2 \leq \dots \leq E.$$

Let  $m < n$ . Then  $A_{mn} = A_n - A_m \geq 0$ . We make use of the generalized Schwarz inequality for the operator  $A_{mn}$ :

$$\|A_{mn} f\|^4 = (A_{mn} f, A_{mn} f)^2 \leq (A_{mn} f, f)(A_{mn}^2 f, A_{mn} f).$$



Since  $0 \leq A_{mn} \leq E$ , we have  $\|A_{mn}\| \leq 1$  and

$$\|A_m f - A_n f\|^4 \leq [(A_n f, f) - (A_m f, f)] \|f\|^2.$$

The numerical sequence  $\{(A_n f, f)\}$  is bounded and nondecreasing and therefore converges. Thus the sequence of vectors  $\{A_n f\}$  is fundamental and by the completeness of  $H$  it converges. The operator  $A$  defined by the equality  $Af = \lim_{n \rightarrow \infty} A_n f$  for any  $f \in H$  is obviously linear and symmetric. ■

## 2.8. The Square Root of a Symmetric Operator

The square of a symmetric operator is always a positive operator, that is,  $(A^2 f, f) = (Af, Af) \geq 0$  for any  $f$ .

The question arises: it is possible to extract the square root of a positive operator?

The following theorem holds.

**THEOREM 8.** *To each symmetric positive operator  $A$  there corresponds a unique positive symmetric square root denoted  $A^{1/2}$  ( $((A)^{1/2})^2 = A$ ). It is the pointwise limit of a sequence of polynomials in  $A$  and therefore commutes with all operators that commute with  $A$ .*

**PROOF:** Without loss of generality we shall assume that  $0 \leq A \leq E$ . Let  $A = E - B$ , ( $0 \leq B \leq E$ ) and  $X = A^{1/2} = E - Y$ . Then solving the equation  $X^2 = (A^{1/2})^2 = A$  is equivalent to solving the equation

$$Y = \frac{1}{2}(B + Y^2).$$

We shall solve this equation by the method of successive approximations:

$$Y_0 = 0, Y_1 = \frac{1}{2}B, \dots, Y_{n+1} = \frac{1}{2}(B + Y_n^2), \quad n \geq 0.$$

We shall show that the sequence  $\{Y_n\}$  converges and its limit is a solution of the desired equation.

We first show that  $Y_n$  is a polynomial in  $B$  with nonnegative real coefficients and that the operator  $Y_n - Y_{n-1}$  has the same form. Let  $n = 1$ . Then these assertions are true. Suppose that they are true also for  $n = m$ . We shall show that they are true for  $n = m + 1$ .

For the operator  $Y_{m+1}$  the assertion is obvious. It follows from its form. Then we have

$$Y_{m+1} - Y_m = \frac{1}{2}(B + Y_m^2) - \frac{1}{2}(B + Y_{m-1}^2) = \frac{1}{2}(Y_m + Y_{m-1})(Y_m - Y_{m-1}).$$

Here we have used the fact that  $Y_m$  and  $Y_{m-1}$  commute, which follows from the induction hypothesis (since they are polynomials in  $B$ ). The right-hand side here is the product of two polynomials in  $B$  with real nonnegative coefficients, and therefore the induction step is complete.

Further if  $B \geq 0$ , then  $B^n \geq 0$ ,  $n = 2, 3, \dots$ . Indeed for  $n = 2k$  we have  $(B^{2k}f, f) = \|B^k f\|^2 \geq 0$  and for  $n = 2k + 1$  we have  $(B^{2k+1}f, f) = (BB^k f, B^k f) = (Bg, g)$  for any vector  $f$ . Therefore  $Y_n \geq 0$  and  $Y_n - Y_{n-1} \geq 0$ . Finally  $\|Y_n\| \leq 1$  for any  $n$ . Indeed for  $n = 0$  this is true and for other  $n$  it can be proved by induction using the equality

$$Y_{n+1} = \frac{1}{2}(B + Y_n^2), \quad n \geq 0.$$

We now apply the theorem just proved on monotonic sequences of operators. We find that the sequence of symmetric operators  $\{Y_n\}$  is monotonic and therefore converges, and that its limit—the operator  $Y$ —satisfies the equation

$$Y = \frac{1}{2}(B + Y^2) \quad \left( \lim_{n \rightarrow \infty} Y_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{2}(B + Y_n^2) \right).$$

Thus we have constructed a symmetric positive operator  $Y$  that is the limit of a sequence of polynomials in  $A$ , and moreover

$$X^2 = A, \quad X = E - Y.$$

We now prove the uniqueness of the square root. Let  $X'$  be a positive symmetric operator such that  $X'^2 = A$ . Since  $X'A = X'(X')^2 = (X')^2 X' = AX'$ , it follows that  $X'$  commutes with all polynomials in the operator  $A$  and also with limits of such operators, in particular with  $X$ . We take square roots  $Z$  and  $Z'$  respectively of  $X$  and  $X'$  constructed as the square root of  $A$  was just constructed. Let  $g = (X - X')f$  for any  $f \in H$ . We have

$$\begin{aligned} \|Zg\|^2 + \|Z'g\|^2 &= (Z^2g, g) + (Z'^2g, g) = (Xg, g) + (X'g, g) \\ &= ((X + X')(X - X')f, g) = ((X^2 - X'^2)f, g) = ((A - A)f, g) = 0. \end{aligned}$$

and therefore  $Zg = Z'g = 0$ . Consequently  $Xg = ZZg = 0$  and  $X'g = Z'Z'g = 0$ . Hence we find that  $\|(X - X')f\|^2 = ((X - X')^2f, f) = ((X - X')g, f) = 0$ , i.e.,  $(X - X')f = 0$ . Since this holds for any  $f \in H$ , we have  $X' = X$ . ■

## 2.9. The Spectral Theorem for a Symmetric Operator on $n$ -Dimensional Space

It is well-known that in solving problems on the reduction of a matrix to Jordan form the question arises of finding its eigenvalues as well as its eigenvectors and conjugate vectors.



If the matrix is given by the numbers  $\{a_{ik}\}$ , then in order to find the eigenvectors and eigenvalues it is necessary to solve the system

$$\sum_{k=1}^n (a_{ik} - \lambda \delta_{ik}) f_k = 0, \quad i = 1, 2, \dots, n.$$

A necessary and sufficient condition for this system to have nonzero solutions is that its determinant vanish. The determinant is a polynomial in  $\lambda$  of degree  $n$  and vanishes for at least one value of  $\lambda$ . Therefore in a finite-dimensional complex vector space a linear operator  $A$  given by a matrix  $\{a_{ik}\}$  (not necessarily Hermitian) has at least one eigenvalue. In the general case this is all that can be said; for example the operator  $y = Af$ , where

$$A = \{a_{ik}\} = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

has a unique eigenvalue  $\lambda = 1$  and it is a simple eigenvalue, i.e., the dimension of the eigenspace corresponding to it is 1.

But if the matrix is Hermitian ( $a_{ik} = \bar{a}_{ki}$ ), or, what is the same,  $(Af, g) = (f, Ag)$ , for any  $f, g \in C^n$ , then there exists an orthonormal system of  $n$  eigenvectors

$$f_i = (f_{i1}, f_{i2}, \dots, f_{in}), \quad i = 1, 2, \dots, n,$$

corresponding to eigenvalues  $\lambda_i$ . Taking this system as a new basis and expanding the vector  $f$  in this basis, we can write

$$(Af, f) = \sum_{i,k=1}^n a_{ik} \bar{g}_i g_k = \sum_{k=1}^n \lambda_k \bar{\alpha}_k \alpha_k = \sum_{k=1}^n \lambda_k |\alpha_k|^2,$$

where  $f = (g_1, \dots, g_n)$ .\* Thus the quadratic form has been reduced to diagonal form. Having generalizations in mind, we shall now obtain this result in a different form.

\*In the new basis of vectors  $f_i$  the vector  $f$  is represented in the form  $f = \sum_{i=1}^n \alpha_i f_i$ .

We enumerate the eigenvalues in increasing order:  $\lambda_1 < \lambda_2 < \dots < \lambda_p$  (here  $p \leq n$ ).

Let  $H_{\lambda_i}$  be the eigenspace of the operator  $A$  corresponding to the eigenvalue  $\lambda_i$ ,  $1 \leq i \leq p$ . As we have seen, if the operator  $A$  is Hermitian, i.e.,  $a_{ik} = \bar{a}_{ki}$ , then for  $\lambda_i \neq \lambda_k$  we have  $H_{\lambda_i} \perp H_{\lambda_k}$ .

In addition  $C^n = H_{\lambda_1} \oplus H_{\lambda_2} \oplus \dots \oplus H_{\lambda_p}$ ,  $p \leq n$ . We set  $H(\lambda_i) = \sum_{j=1}^i \oplus H_{\lambda_j}$ . Then we can write

$$H = \sum_{i=1}^p \oplus [H(\lambda_i) \ominus H(\lambda_{i-1})], \quad H(\lambda_0) = \{0\}.$$

(Throughout this expression the equalities are understood in the sense that every vector of  $H = C^n$  has a unique expansion over the components on the right-hand side.)

The subspaces  $H(\lambda_i)$  form an increasing sequence

$$\{0\} = H(\lambda_0) \subset H(\lambda_1) \subset \dots \subset H(\lambda_p) = H.$$

We denote by  $E(\lambda_i)$  the orthogonal projection on the subspace  $H(\lambda_i)$ , and by Proposition 5 of Sec. 4.2.5 we find that

$$0 = E(\lambda_0) \leq E(\lambda_1) \leq \dots \leq E(\lambda_{p-1}) \leq E(\lambda_p) = E.$$

Let  $E_{\lambda_i}$  be the projection on  $H(\lambda_i) \ominus H(\lambda_{i-1})$ . By Proposition 4 of Sec. 4.2.5 we have  $E_{\lambda_i} = E(\lambda_i) - E(\lambda_{i-1})$ . Therefore in terms of projection operators the decomposition

$$H = \sum_{i=1}^p [H(\lambda_i) \ominus H(\lambda_{i-1})],$$

can be written

$$E = \sum_{i=1}^p [E(\lambda_i) - E(\lambda_{i-1})] = \sum_{i=1}^p E_{\lambda_i}, \quad E(\lambda_0) = \{0\},$$

(here  $E$  is the identity operator, i.e., the projection on all of  $H$ ). Since  $E_{\lambda_i}$  is the projection on the eigenspace  $H_{\lambda_i}$ , we have  $AE_{\lambda_i}f = \lambda_i E_{\lambda_i}f$  for any  $f \in H$ , i.e.,  $AE_{\lambda_i} = \lambda_i E_{\lambda_i}$ . Hence

$$\begin{aligned} A \sum_{i=1}^p [E(\lambda_i) - E(\lambda_{i-1})] &= A \sum_{i=1}^p E_{\lambda_i} = \sum_{i=1}^p \lambda_i E_{\lambda_i} \\ &= \sum_{i=1}^p \lambda_i [E(\lambda_i) - E(\lambda_{i-1})]. \end{aligned}$$



Since

$$E_{\lambda_i} \cdot E_{\lambda_k} = E_{\lambda_k} E_{\lambda_i} = \begin{cases} 0 & \text{for } i \neq k, \\ E_{\lambda_k} & \text{for } i = k, \end{cases}$$

we have the formula

$$A^m = \sum_{i=1}^p \lambda_i^m E_{\lambda_i}$$

for the operator  $A$ . Consequently any polynomial  $p(A)$  of degree  $n$  in the operator  $A$ ,

$$p(A) = \alpha_0 E + \alpha_1 A + \dots + \alpha_n A^n,$$

where  $\alpha_i$  are numbers, can be written in the form

$$p(A) = \sum_{i=1}^p p(\lambda_i) E_{\lambda_i} = \sum_{i=1}^p p(\lambda_i) [E(\lambda_i) - E(\lambda_{i-1})].$$

We shall verify below that similar formulas hold in the case of an arbitrary symmetric operator in Hilbert space. The formula

$$A = \sum_{i=1}^p \lambda_i [E(\lambda_i) - E(\lambda_{i-1})]$$

is called the *spectral decomposition* of the Hermitian operator  $A$  in  $C^n$  and can be rewritten in the form

$$(Af, g) = \sum_{i=1}^p \lambda_i (\Delta_i E(\lambda) f, g) \quad \text{for any } f, g \in H,$$

$$(\Delta_i E(\lambda) f, g) = (E(\lambda_i) f, g) - (E(\lambda_{i-1}) f, g).$$

Thus we have proved the spectral theorem for a Hermitian operator on a finite-dimensional space.

**THEOREM 9.** *To every Hermitian operator  $A$  on the finite-dimensional space  $C^n$  there corresponds a certain family of projections  $E(\lambda_i)$ , where  $i = 1, 2, \dots, p \leq n$ , and  $\lambda_1, \dots, \lambda_p$  are the eigenvalues of the operator  $A$ , having the properties:*

- $E(\lambda_i) \leq E(\lambda_j)$  if  $\lambda_i \leq \lambda_j$ .
- $E(\lambda_p) = E$ ,  $E(\lambda_0) = \{0\}$ .
- The operator  $A$  can be represented using  $E(\lambda_i)$  in the form

$$A = \sum_{i=1}^p \lambda_i [E(\lambda_i) - E(\lambda_{i-1})] = \int_{-\infty}^{+\infty} \lambda dE(\lambda)$$

where the integral is a formal notation for the sum in the middle.

## 2.10. Completely Continuous Operators. The Spectral Theorem

The simplest generalization of the spectral theorem for the finite-dimensional case is its analog for completely continuous operators.

Let  $A$  be a completely continuous symmetric operator in a Hilbert space  $H$ . Then, as has been shown,

$$\sup_{\|f\|=1} |(Af, f)| = \sup_{\|f\|=1} \|Af\| = \|A\|.$$

Let  $\{g_n\}$  be a sequence of elements such that  $\|g_n\| = 1$  and  $|(Ag_n, g_n)| \rightarrow \|A\|$  as  $n \rightarrow \infty$ . We assume that the enumeration is such that  $(Ag_n, g_n)$  itself converges to some real number  $\lambda_1$ ;  $\lambda_1 = \|A\|$  or  $\lambda_1 = -\|A\|$ . Consider the quadratic trinomial

$$\|Ag_n\|^2 - 2\lambda_1(Ag_n, g_n) + \lambda_1^2\|g_n\|^2 = \|Ag_n - \lambda_1 g_n\|^2 \geq 0.$$

Since  $\|Ag_n\|^2 \leq \|A\|^2 = \lambda_1^2$ ,  $(Ag_n, g_n) \rightarrow \lambda_1$ , and  $\|g_n\|^2 = 1$ , it follows that  $\|Ag_n - \lambda_1 g_n\| \rightarrow 0$  as  $n \rightarrow \infty$ , and therefore  $Ag_n - \lambda_1 g_n \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $A$  is a completely continuous operator, the sequence  $\{Ag_n\}$  contains a convergent subsequence  $\{Ag_{n_k}\}$ ; since  $Ag_{n_k} - \lambda_1 g_{n_k} \rightarrow 0$ , the sequence  $\{g_{n_k}\}$  also converges to some limit  $f_1$ . We have  $Af_1 = \lim Ag_{n_k}$ ,  $\|f_1\| = \lim \|g_{n_k}\| = 1$ , and consequently  $Af_1 = \lambda_1 f_1$ . In addition

$$\begin{aligned} |(Af_1, f_1)| &= |(\lambda_1 f_1, f_1)| = |\lambda_1| = \|A\|, \\ \|Af_1\| &= \|\lambda_1 f_1\| = |\lambda_1| = \|A\|. \end{aligned}$$

Thus every completely continuous symmetric operator  $A \neq 0$  has at least one nonzero eigenvalue  $\lambda_1$ . This eigenvector is a solution of the following extremal problem: find a vector  $\varphi$  with  $\|\varphi\| = 1$  for which  $|(A\varphi, \varphi)|$  attains its maximum. This eigenvector will be denoted  $f_1$ . We now try to find other eigenvectors of the operator  $H$  orthogonal to  $f_1$ . Consider the decomposition  $H = H_1 \oplus V_1$ , where  $V_1 = \{f_1\}$  is the subspace spanned by the vector  $f_1$ . The subspace  $H_1$  is invariant with respect to the operator  $A$ , i.e.,  $AH_1 \subset H_1$ . Indeed,  $(Af, f_1) = (f, Af_1) = (f, \lambda_1 f_1) = \lambda_1(f, f_1) = 0$  for any  $f \in H_1$ . Therefore the image of the vector also belongs to  $H_1$ .

The operator  $A$ , regarded as an operator on  $H_1$ , is completely continuous and symmetric. Consequently in  $H_1$  there exists an eigenvector  $f_2$  with  $\|f_2\| = 1$  and a corresponding eigenvalue  $\lambda_2$  whose absolute value is the largest value of  $|(Af, f)|$  or  $\|Af\|$  under the conditions that  $f \in H_1$  and  $\|f\| = 1$ . This process can be continued.



Thus we obtain an infinite system  $\{f_n\}$  of eigenvectors; this system is orthonormal. By the construction  $|\lambda_1| \geq |\lambda_2| \geq \dots$ . We shall show that  $\lambda_n \rightarrow 0$  as  $n \rightarrow \infty$ . Suppose the contrary. Then the sequence  $\left\{\frac{1}{\lambda_n} f_n\right\}$  is bounded and the sequence of its images  $\{f_n\}$  under the mapping  $A$  will contain a convergent subsequence, which is impossible, since  $\|f_i - f_j\|^2 = 2$  if  $i \neq j$ . Let  $f$  be any vector in  $H$ . We set

$$h_n = f - \sum_{i=1}^n (f, f_i) f_i.$$

Since each  $h_n$  belongs to the subspace  $H_n$  formed by the vectors orthogonal to the vectors  $f_1, f_2, \dots, f_n$ , we have  $\|Ah_n\| \leq |\lambda_{n+1}| \cdot \|h_n\|$ ; in addition

$$\|h_n\|^2 = \|f\|^2 - \sum_{i=1}^n |(f, f_i)|^2 \leq \|f\|^2, \quad \lambda_{n+1} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and so  $Ah_n = Af - \sum_{i=1}^n (f, f_i) Af_i \rightarrow 0$  as  $n \rightarrow \infty$ , which can be written as follows:

$$Af = \sum_{i=1}^{\infty} \lambda_i (f, f_i) f_i = \sum_{i=1}^{\infty} (f, Af_i) f_i = \sum_{i=1}^{\infty} (Af, f_i) f_i.$$

Such a series converges to a finite sum if all the  $\lambda_i$  from some index on are zero. The sequence (finite or infinite) of numbers  $\lambda_i \neq 0$  contains each nonzero eigenvalue of the operator  $A$  a number of times equal to its multiplicity, since otherwise there would be an eigenvector  $\varphi$  corresponding to an eigenvalue  $\lambda \neq 0$  orthogonal to all  $f$ . Then we would have

$$0 \neq A\varphi = \lambda\varphi = \sum_{i=1}^{\infty} \lambda_i (\varphi, f_i) f_i = 0,$$

and thereby we would reach a contradiction. It also follows from what has been said that any eigenvalue  $\lambda \neq 0$  of the operator  $A$  has finite multiplicity.

Otherwise we would need to examine again the set  $\left\{\frac{1}{\lambda}\varphi_k\right\}$ , where  $A\varphi_k = \lambda\varphi_k$ . We write the formula  $Af = \sum_{i=1}^{\infty} \lambda_i (f, f_i) f_i$  in the form

$$Af = \int_{-\infty}^{\infty} \lambda dE(\lambda)f, \quad E(\lambda)f = \begin{cases} \sum_{\lambda_k \leq \lambda} (f, f_k) f_k & \text{for } \lambda < 0, \\ f - \sum_{\lambda_k > \lambda} (f, f_k) f_k & \text{for } \lambda \geq 0. \end{cases}$$

The integral just written is a Stieltjes integral. We remark that  $E(\lambda)$ , being a function of  $\lambda$ , is constant between any two successive eigenvalues of the operator  $A$ , equal to 0 for values of  $\lambda$  less than all the eigenvalues, and equal to the identity  $E$  for values of  $\lambda$  larger than all of the eigenvalues. When  $\lambda$  passes through the eigenvalue  $\lambda_p$  as it varies, the operator  $E(\lambda)$  undergoes a jump equal to  $\hat{E}(\lambda_p) = E(\lambda_p) - E(\lambda_p - 0)$ , and  $\hat{E}(\lambda_p)$  is the projection on the eigenspace corresponding to the given eigenvalue  $\lambda_p$ . Thus

$$\begin{aligned}(E(\lambda_p) - E(\lambda_p - 0))f &= \hat{E}(\lambda_p)f = \sum_{\lambda_k = \lambda_p} (f, f_k) f_k \quad \text{for } \lambda_p \neq 0, \\ \hat{E}(0)f &= f - \sum_{\lambda_k \neq 0} (f, f_k) f_k.\end{aligned}$$

Thus we have proved the spectral theorem for a symmetric completely continuous operator.

**THEOREM 10.** *To every completely continuous symmetric operator  $A$  on  $H$  there corresponds a family of projections  $E(\lambda)$  depending on the real parameter  $\lambda$  and having the properties:*

- a)  $E(\lambda) \leq E(\mu)$  for  $\lambda \leq \mu$ ;
- b)  $E(\lambda + 0) = E(\lambda)$ ;
- c)  $E(\lambda) = 0$  for  $\lambda$  less than the smallest eigenvalue of the operator, and  $E(\lambda) = E$  for  $\lambda$  larger than the largest eigenvalue of the operator. Using  $E(\lambda)$  the operator can be represented in the form

$$A = \int_{-\infty}^{+\infty} \lambda dE(\lambda).$$

### 2.11. The Spectral Theorem for a Bounded Symmetric Operator

Let  $A$  be a symmetric operator. With an arbitrary polynomial having real coefficients

$$p(\lambda) = a_0 + a_1 \lambda + \cdots + a_n \lambda^n$$

we associate the symmetric operator

$$p(A) = a_0 E + a_1 A + \cdots + a_n A^n.$$

This correspondence is obviously homogeneous, additive, and multiplicative; this means that the polynomials  $cp(\lambda)$ ,  $p(\lambda) + q(\lambda)$ , and  $p(\lambda)q(\lambda)$  correspond



to the operators  $cp(A)$ ,  $p(A) + q(A)$ , and  $p(A)q(A)$ . Furthermore this correspondence is of positive type, i.e., if  $p(\lambda) \geq 0$  for  $m \leq \lambda \leq M$ , where  $M$  and  $m$  are upper and lower bounds of the operator  $A$ , then the operator  $p(A) \geq 0$ .

To prove this we represent  $p(\lambda)$  in the form

$$p(\lambda) = d \prod_i (\lambda - \alpha_i) \prod_j (\beta_j - \lambda_j) \prod_k [(\lambda - \gamma_k)^2 + \delta_k^2],$$

where  $d \geq 0$ ,  $\alpha_i \leq m$ ,  $\beta_j \geq M$ , and the factors in square brackets correspond to complex conjugate pairs of roots and, when  $\delta_k = 0$ , to the roots between  $m$  and  $M$ . By the fact that the polynomial  $p(\lambda)$  is nonnegative, these last roots are all of even multiplicity. If we substitute the operator  $A$ , we obtain a representation of the operator  $p(A)$  in the form of a composition of commuting operators, all positive. It follows from this that  $p(A)$  is a positive operator. Indeed, if  $A_1$  and  $A_2$  are commuting positive symmetric operators, then  $A_1 A_2$  is a positive symmetric operator:

$$(A_1 A_2 f, f) = (A_1 A_2^{\frac{1}{2}} A_2^{\frac{1}{2}} f, f) = (A_2^{\frac{1}{2}} A_1 A_2^{\frac{1}{2}} f, f) = (A_1 A_2^{\frac{1}{2}} f, A_2^{\frac{1}{2}} f) \geq 0.$$

The symmetry is obvious. It even follows from this that the inequality  $A_1 \geq A_2$  is preserved if both sides are composed with the same positive symmetric operator  $C$  commuting with both  $A_1$  and  $A_2$ :

$$A_1 C = C A_1 \geq C A_2 = A_2 C.$$

Thus we can say that if two polynomials are such that  $p(\lambda) \geq q(\lambda)$  for  $m \leq \lambda \leq M$ , then  $p(A) \geq q(A)$ .

We now take up the problem of extending this correspondence to functions that are not polynomials. We denote by the letter  $C$  the class consisting of the continuous real-valued functions  $\{\varphi\}$  on the closed interval  $[m, M]$  together with the piecewise continuous functions (which we denote  $\{\psi\}$ ) that are limits of monotonically decreasing sequences of continuous functions  $\{\varphi_n\}$  convergent at each point. The following result holds.

**LEMMA 1.** *For every function  $\psi(\lambda) \in C$ ,  $m \leq \lambda \leq M$ , one can construct an infinite sequence of polynomials  $p_n(\lambda)$ ,  $m \leq \lambda \leq M$ , monotonically decreasing and converging to the function  $\psi(\lambda)$  at every point.*

**PROOF:** By definition of the class  $C$  there exists a sequence  $\{\varphi_n\}$  that is monotonically decreasing and converges to the function  $\psi(\lambda)$  at every point. Let  $n$  be fixed, and approximate the continuous function  $\varphi_n(\lambda) +$

$\frac{3}{2^{n+2}}$  within  $\frac{1}{2^{n+2}}$  by a polynomial  $p_n(\lambda)$ , for  $m \leq \lambda \leq M$ . We have  $\left| p_n(\lambda) - \left[ \varphi_n(\lambda) + \frac{3}{2^{n+2}} \right] \right| \leq \frac{1}{2^{n+2}}$ . Consequently at each point  $\lambda$  we have  $\frac{1}{2^{n+1}} \leq p_n(\lambda) - \varphi_n(\lambda) \leq \frac{1}{2^n}$ , and therefore the polynomial  $p_n(\lambda)$  tends to  $\psi(\lambda)$  at every point  $\lambda$  along with  $\varphi_n(\lambda)$ . The sequence  $p_n(\lambda)$  is monotonically decreasing:

$$p_{n+1}(\lambda) \leq \varphi_{n+1}(\lambda) + \frac{1}{2^{n+1}} \leq \varphi_n(\lambda) + \frac{1}{2^{n+1}} \leq p_n(\lambda),$$

which was to be proved. ■

We now construct the operators  $p_n(A)$ . They are symmetric, decrease monotonically, and are bounded below by the operator  $\alpha E$ , where  $\alpha = \inf_{m \leq \lambda \leq M} \psi(\lambda)$ . According to Theorem 7 of Sec. 4.2.6 they converge pointwise to some operator, which we take as  $\psi(A)$  by definition. The operator  $\psi(A)$  is independent of the particular choice of the sequence  $\{p_n(\lambda)\}$ : if  $\{q_n(\lambda)\}$  is another sequence of the same type, then  $\lim_{n \rightarrow \infty} p_n(A) = \lim_{n \rightarrow \infty} q_n(A)$ . In fact, for any integer  $r$  the inequalities

$$p_s(\lambda) \leq q_r(\lambda) + \frac{1}{r}, \quad q_s(\lambda) \leq p_r(\lambda) + \frac{1}{r}$$

are fulfilled for sufficiently large  $s$  at each point  $\lambda$ , and by the Heine-Borel theorem, for all  $\lambda \in [m, M]$ .

Indeed the inequalities hold in a neighborhood of each point  $\lambda$  by the continuity of the functions occurring in them. Choosing a finite cover of the closed interval  $[m, M]$ , we obtain the required result. Then  $p_s(A) \leq q_r(A) + \frac{E}{r}$  and  $q_s(A) \leq p_r(A) + \frac{E}{r}$ . Passing to the limit in these inequalities first as  $s \rightarrow \infty$ , then as  $r \rightarrow \infty$ , we find that

$$\lim_{n \rightarrow \infty} p_n(A) = \lim_{n \rightarrow \infty} q_n(A).$$

By the same reasoning it is easy to show that if  $\psi_1(\lambda) \geq \psi_2(\lambda)$ , for  $m \leq \lambda \leq M$ , then  $\psi_1(A) \geq \psi_2(A)$ . Thus the correspondence between operators and functions of class  $C$ , as one can easily deduce, is additive, homogeneous, and multiplicative (the last three properties follow from properties of the limit).

It is obvious that the correspondence between the symmetric operator  $A$  and functions can be extended to a wider class than  $C$ , namely the class  $C_1$  of functions that can be represented as differences of functions of class  $C$ . The



properties of monotonicity, additivity, homogeneity, and multiplicativeness can be preserved.

Among the "functions" of a symmetric operator  $A$  that have just been defined, there are projections; they correspond to functions  $e(\lambda)$  that assume only the values 0 and 1. Here obviously  $[e(\lambda)]^2 = e(\lambda)$ , and therefore  $[e(A)]^2 = e(A)$ . The operator  $e(A)$  is symmetric, being the limit of symmetric operators, and so since  $[e(A)]^2 = e(A)$ , it is, as we know, a projection.

Consider in particular the function  $e_\mu(\lambda)$  depending on the parameter  $\mu$  and assuming the values 1 and 0 respectively for  $\lambda \leq \mu$  and  $\lambda > \mu$ , while  $e_\mu(\lambda) = 0$  for  $\mu < m$  and  $e_\mu(\lambda) = 1$  for  $\mu \geq M$ . The function  $e_\mu(\lambda)$  belongs to  $C$  and consequently corresponds to a projection  $e_\mu(A)$  that we shall denote by  $E(\mu)$ . Since

$$e_\mu(\lambda)e_\nu(\lambda) = e_\mu(\lambda), \quad \text{for } \mu < \nu,$$

we obtain by the multiplicative property of the correspondence

$$E(\mu) \cdot E(\nu) = E(\nu)E(\mu) = E(\mu).$$

Since  $e_\mu(\lambda) \leq e_\nu(\lambda)$  for  $\mu \leq \nu$ , we have also  $E(\mu) \leq E(\nu)$ ; since by definition  $e_\mu(\lambda) = 0$  for  $\mu < m$  and  $e_\mu(\lambda) = 1$  for  $\mu \geq M$  in the interval  $m \leq \lambda \leq M$ , we have  $E(\mu) = 0$  for  $\mu < m$  and  $E(\mu) = E$  for  $\mu \geq M$ .

The function  $E(\mu)$ , regarded as a function of the parameter  $\mu$ , is continuous from the right. Indeed, fix some value of  $\mu$  and take a decreasing sequence of polynomials  $p_n(\lambda)$  tending to  $e_\mu(\lambda)$  for  $\lambda \in [m, M]$ , and assume that  $p_n(\lambda) \geq e_{\mu + \frac{1}{n}}(\lambda)$ . Then  $p_n(A) \geq E\left(\mu + \frac{1}{n}\right) \geq E(\mu)$ . Since  $p_n(A) \rightarrow E(\mu)$  as  $n \rightarrow \infty$ , it follows that  $E\left(\mu + \frac{1}{n}\right) \rightarrow E(\mu)$  as  $n \rightarrow \infty$ . Therefore if  $\varepsilon \rightarrow 0$ , so that  $0 < \varepsilon < 1/n$ , then  $E\left(\mu + \frac{1}{n}\right) \geq E(\mu + \varepsilon) \geq E(\mu)$ . Consequently  $E(\mu + \varepsilon) \rightarrow E(\mu)$  as  $\varepsilon \rightarrow 0$ .

We now obtain an integral representation for the operator  $A$ . We remark that if  $\mu < \nu$ , then obviously we have the inequalities

$$\mu[e_\nu(\lambda) - e_\mu(\lambda)] \leq \lambda[e_\nu(\lambda) - e_\mu(\lambda)] \leq \nu[e_\nu(\lambda) - e_\mu(\lambda)],$$

so that we can write

$$\mu(E(\nu) - E(\mu)) \leq A(E(\nu) - E(\mu)) \leq \nu \cdot (E(\nu) - E(\mu)).$$

Let  $\mu_0 < m < \mu_1 < \mu_2 < \dots < \mu_{n-1} < M \leq \mu_n$ . Writing the inequalities given above for  $\mu = \mu_{k-1}$  and  $\nu = \mu_k$ ,  $k = 1, 2, \dots, n$  and

summing, we obtain

$$\begin{aligned} \sum_{k=1}^n \mu_{k-1} (E(\mu_k) - E(\mu_{k-1})) &\leq A \sum_{k=1}^n (E(\mu_k) - E(\mu_{k-1})) \\ &\leq \sum_{k=1}^n \mu_k (E(\mu_k) - E(\mu_{k-1})). \end{aligned}$$

The second of these sums is equal to  $A(E(\mu_n) - E(\mu_0)) = A(E - 0) = A$ , and for  $\max_k (\mu_k - \mu_{k-1}) \leq \varepsilon$  the difference between the first and last terms in these inequalities is at most  $\varepsilon E$ . Therefore  $\left\| A - \sum_{k=1}^n \lambda_k (E(\mu_k) - E(\mu_{k-1})) \right\| \leq \varepsilon$ , where  $\lambda_k$  is contained between  $\mu_k$  and  $\mu_{k-1}$ .

If the number  $n$  of partial intervals  $(\mu_{k-1}, \mu_k)$  is increased to infinity in such a way that the largest of their lengths tends to zero, then the sums  $\sum_{k=1}^n \lambda_k (E(\mu_k) - E(\mu_{k-1}))$  will tend in norm to the operator  $A$ . Since  $E(\lambda)$  is constant as a function of  $\lambda$  for  $\lambda \geq M$  and for  $\lambda < m$ , the result just obtained can be written in a form analogous to an ordinary Stieltjes integral

$$A = \int_{-\infty}^{+\infty} \lambda dE(\lambda) = \int_{m-0}^M \lambda dE(\lambda).$$

At the point  $m$  we take the function  $E$  to be equal to  $E(m-0) = 0$ .

As noted the approximating sums for the integral tend to the operator  $A$  in norm, and therefore for any  $f, g \in H$  the relation

$$(Af, g) = \int_{m-0}^M \lambda d(E(\lambda)f, g)$$

will hold (since weak convergence is implied by norm convergence). The left-hand side of this equality is defined independently of  $E(\lambda)$ . According to theorems on the ordinary Stieltjes integral the numerical function  $(E(\lambda)f, g)$  is determined up to a constant term by the equality above at points of continuity and also at  $\lambda = m-0$  and  $\lambda = M$ . Since this function is continuous from the right and assumes the value  $(f, g)$  at the point  $M$ , it is uniquely determined everywhere, from which it follows that the family  $E(\lambda)$  is uniquely determined by the operator  $A$ . Thus we have proved the spectral theorem for a bounded symmetric linear operator.

**THEOREM 11.** *To every symmetric operator  $A$  in  $H$  having greatest lower bound  $m$  and least upper bound  $M$  one can assign a unique family of*



projections  $E(\lambda)$  depending on the real parameter  $\lambda$ ,  $\lambda \in [m, M]$ , and having the following properties:

a)  $E(\lambda) \leq E(\mu)$  for  $\lambda \leq \mu$ ;

b)  $E(\lambda + 0) = E(\lambda)$ ;

c)  $E(\lambda) = 0$  for  $\lambda < m$  and  $E(\lambda) = E$  for  $\lambda \geq M$ . Using  $E(\lambda)$  the operator  $A$  can be represented in the form\*

$$A = \int_{-\infty}^{\infty} \lambda dE(\lambda) = \int_{m-0}^M \lambda dE(\lambda).$$

#### EXAMPLE

Consider the operator of multiplication by the independent variable in the space  $L^2[0, 1]$ . Here  $Af(x) = xf(x)$ . We have  $A^2f(x) = x^2f(x)$ , and for any polynomial  $p(x)$  we have  $p(A)f(x) = p(x)f(x)$ . For any function  $\psi(x)$  of class  $C$  we obtain  $\psi(A)f(x) = \psi(x)f(x)$ .

We need to calculate the projection corresponding to the function

$$e_\mu(x) = \begin{cases} 1 & \text{if } x \leq \mu, \\ 0 & \text{if } x > \mu. \end{cases}$$

This function belongs to the class  $C$ , and in particular  $E_\mu f(x) = e_\mu(x)f(x)$ , i.e., the projection  $E(\mu)$  of the spectral family acts on the space  $L^2[0, 1]$  as the operator of multiplication by the function  $e_\mu(x)$ . We obviously have

$$A = x = \int_0^1 \lambda de_\lambda(x), \quad 0 \leq x \leq 1.$$

## 2.12. The Spectral Theorem for a Unitary Operator

By analogy with the proof of the spectral theorem for bounded symmetric operators, one can also obtain a spectral theorem for unitary operators,

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\*It is obvious that the following representations are valid for a polynomial  $p(\lambda)$  and even for any continuous function  $u(\lambda)$ :

$$p(A) = \int_{m-0}^M p(\lambda) dE(\lambda), \quad u(A) = \int_{m-0}^M u(\lambda) dE(\lambda).$$

which will now be defined. Essential use will also be made of the spectral theorem for unitary operators in the proof of the spectral theorem for unbounded symmetric operators.

**DEFINITION 4.** A linear operator  $U$  in a Hilbert space  $H$  is called an *isometry* if it preserves the value of the inner product, i.e., if

$$(Uf, Ug) = (f, g) \quad \text{for any } f, g \in H.$$

If in addition  $U$  maps  $H$  onto all of  $H$ , then  $U$  is called *unitary*.

**PROPOSITION 6.** A bounded linear operator  $U$  mapping a Hilbert space  $H$  into itself is unitary if and only if  $U^* = U^{-1}$ .

**PROOF:** If  $U$  is unitary, then the relation

$$\|Uf\| = \|f\|$$

implies that the equation  $Uf = 0$  has no solutions except  $f = 0$ , from which it follows that there exists a bounded inverse  $U^{-1}$  defined\* on all of  $H$ :

$$D_U = R(U) = H.$$

Let  $g = U^{-1}h$ . Then for any  $h \in H$  we have  $(Uf, h) = (f, U^{-1}h)$ , i.e.,  $U^* = U^{-1}$ . Conversely the condition  $U^* = U^{-1}$  implies that the scalar product is invariant:

$$(Uf, Ug) = (f, U^*Ug) = (f, U^{-1}Ug) = (f, g).$$

Since the operator  $U$  is bounded, the operator  $U^*$  is defined on all of the space  $H$  and so we have

$$R(U) = D_{U^{-1}} = D_{U^*} = H. \blacksquare$$

#### EXAMPLE

In the space  $H = L^2(-\infty, +\infty)$  the operator  $U$  acts on any vector  $x(t) \in L^2(-\infty, +\infty)$  as follows:

$$Ux(t) = x(t + a),$$

---

\*The symbols  $D_U$  and  $R(U)$  respectively denote the domain of definition and range of values of the operator  $U$ .



where  $a$  is an arbitrary real number. The operator  $U$  is unitary.

In a finite-dimensional space every operator with zero kernel produces a mapping onto the entire space. Therefore any linear isometry in a finite-dimensional space is unitary. In the general case this is not true. In the space  $l^2$  let the operator  $U$  act on any vector  $\xi = (\xi_1, \xi_2, \dots, \xi_n, \dots)$  as follows:  $U\xi = \xi_0 = (0, \xi_1, \xi_2, \dots)$ . The operator  $U$  is an isometry, but is not unitary.

There is a spectral theorem for unitary operators analogous to Theorem 11 for bounded symmetric operators. Let us prove it. We begin by associating with a trigonometric polynomial

$$P(e^{i\varphi}) = \sum_{k=-m}^n c_k e^{ik\varphi}$$

an operator

$$P(U) = \sum_{k=-m}^n c_k U^k,$$

where  $U$  is a unitary operator and the coefficients  $c_k$  can be any complex numbers. Obviously this correspondence is homogeneous, additive, and multiplicative. To the conjugate polynomial

$$\overline{P(e^{i\varphi})} = \sum_{k=-m}^n \bar{c}_k e^{-ik\varphi}.$$

there corresponds the operator

$$T = \sum_{k=-m}^n \bar{c}_k U^{-k},$$

that is the adjoint of the operator  $P(U)$ :

$$(P(U))^* = \left( \sum_{k=-m}^n c_k U^k \right)^* = \sum_{k=-m}^n \bar{c}_k U^{-k} = T.$$

From the representations for  $P(U)$  and  $T$  we find that the operator  $P(U)$  is symmetric if and only if  $n = m$  and  $c_k = \bar{c}_{-k}$ , i.e., if and only if the polynomial  $P(e^{i\varphi})$  assumes only real values.

The following lemma makes it possible to assert that the correspondence thus introduced is of positive type, i.e., if

$$P(e^{i\varphi}) \geq 0 \quad (0 \leq \varphi \leq 2\pi),$$

then

$$P(U) \geq 0.$$

LEMMA 2. Every trigonometric polynomial  $P(e^{i\varphi}) \geq 0$  can be represented as the square of the absolute value of some other trigonometric polynomial  $Q(e^{i\varphi})$ :

$$P(e^{i\varphi}) = |Q(e^{i\varphi})|^2.$$

PROOF: Since  $P(e^{i\varphi})$  is real-valued, we have

$$P(e^{i\varphi}) = \sum_{k=-n}^n c_k e^{ik\varphi}, \quad \text{where } c_k = \bar{c}_{-k}.$$

We write the representation

$$P(e^{i\varphi}) = e^{-in\varphi} \cdot (c_{-n} + c_{-n+1} e^{i\varphi} + \cdots + c_n e^{i2n\varphi}) = e^{-in\varphi} \cdot M(e^{i\varphi}),$$

where

$$M(z) = z^{2n} \overline{M\left(\frac{1}{\bar{z}}\right)}, \quad z \neq 0.$$

It suffices to prove the lemma for polynomials  $P(e^{i\varphi}) > 0$  on  $0 \leq \varphi \leq 2\pi$ , since in the general case we can add  $\varepsilon > 0$  to  $P(e^{i\varphi})$  and then pass to the limit as  $\varepsilon \rightarrow 0$ . By the assumptions made the polynomial  $M(z)$  has no zeros on the circle  $|z| = 1$ .

From the relation for  $P(e^{i\varphi})$  we conclude that if  $z_k$  is a zero of  $M(z)$  lying inside the circle, then  $\frac{1}{\bar{z}_k}$  is also a zero lying outside the circle  $|z| = 1$  of the same multiplicity as  $z_k$  and conversely.

Consequently

$$\begin{aligned} M(z) &= c_n \prod_k (z - z_k)^{l_k} \left(z - \frac{1}{\bar{z}_k}\right)^{l_k} = c_n \prod_k (z - z_k)^{l_k} \frac{z^{l_k}}{\bar{z}_k^{l_k}} \left(\bar{z}_k - \frac{1}{z}\right)^{l_k} \\ &= (-1)^n c_n z^n \prod_k (z - z_k)^{l_k} \left(\frac{1}{z} - \bar{z}_k\right)^{l_k} \frac{1}{\bar{z}_k^{l_k}} = c z^n \prod_k (z - z_k)^{l_k} \left(\frac{1}{z} - \bar{z}_k\right)^{l_k}. \end{aligned}$$

Therefore

$$\begin{aligned} P(e^{i\varphi}) &= e^{-in\varphi} M(e^{i\varphi}) = c \prod_k (e^{i\varphi} - z_k)^{l_k} (e^{-i\varphi} - \bar{z}_k)^{l_k} \\ &= c \prod_k (e^{i\varphi} - z_k)^{l_k} \overline{\prod_k (e^{i\varphi} - z_k)^{l_k}}. \end{aligned}$$



The constant  $c$  is positive, since  $P(e^{i\varphi}) > 0$ . Consequently the polynomial

$$Q(e^{i\varphi}) = \sqrt{c} \prod_k (e^{i\varphi} - z_k)^{l_k}$$

is the one sought. ■

By the lemma just proved a positive polynomial can be represented in the form

$$P(e^{i\varphi}) = Q(e^{i\varphi}) \overline{Q(e^{i\varphi})},$$

and therefore

$$P(U) = Q(U)Q(U)^*$$

so that for any  $f \in H$

$$(P(U)f, f) = (Q(U)Q(U)^*f, f) = (Q(U)^*f, Q(U)^*f) \geq 0.$$

The correspondence just established between trigonometric polynomials and operators can be extended to more general functions of period  $2\pi$  so as to preserve linearity, multiplicativeness, and monotonicity. We introduce the class of functions  $C$  that consists of the continuous real-valued functions on the unit circle  $\{\eta(e^{i\varphi})\}$  together with the piecewise-continuous functions  $\{\Psi(e^{i\varphi})\}$  on  $0 \leq \varphi \leq 2\pi$  that are limits of monotonically decreasing sequences  $\{\eta_n(e^{i\varphi})\}$  that converge at each point. In analogy with the proof of Lemma 1 for bounded symmetric operators we prove the following lemma.

LEMMA 3. *For every function  $\Psi(e^{i\varphi}) \in C$  one can construct an infinite sequence of trigonometric polynomials  $\{P_n(e^{i\varphi})\}_1^\infty$  that is monotonically decreasing and converges at each point to the function  $\Psi(e^{i\varphi})$ .*

The sequence of symmetric operators  $P_n(U)$  decreases monotonically and is bounded below by the operator  $\alpha E$ , where  $\alpha = \inf_{0 \leq \varphi \leq 2\pi} \Psi(e^{i\varphi})$ . Consequently by Theorem 7 of Sec. 4.2.7 they converge pointwise to some symmetric operator, which we take as  $\Psi(U)$ . As was done in Sec. 4.2.11 for bounded symmetric operators, it can be shown that the operator  $\Psi(U)$  is independent of the particular choice of the sequence  $P_n(e^{i\varphi})$ .

Thus the correspondence between functions and operators extends to the entire class  $C$  preserving the properties of this correspondence that hold for trigonometric polynomials. Consider the class  $C_1$  of functions that can be represented as differences of functions of class  $C$ . To the function

$$\Psi_3(e^{i\varphi}) = \Psi_1(e^{i\varphi}) - \Psi_2(e^{i\varphi})$$

we assign the operator

$$\Psi_3(U) = \Psi_1(U) - \Psi_2(U).$$

The operator  $\Psi_3(U)$  is unambiguously defined. Indeed, suppose  $\Psi_3(e^{i\varphi})$  is representable in another way as a difference of two elements of  $C$ :

$$\Psi_3(e^{i\varphi}) = \Psi_4(e^{i\varphi}) - \Psi_5(e^{i\varphi}).$$

Then from the identity

$$\Psi_1(e^{i\varphi}) - \Psi_2(e^{i\varphi}) = \Psi_4(e^{i\varphi}) - \Psi_5(e^{i\varphi})$$

follows the identity

$$\Psi_1(e^{i\varphi}) + \Psi_5(e^{i\varphi}) = \Psi_4(e^{i\varphi}) + \Psi_2(e^{i\varphi}),$$

both sides of which belong to the class  $C$ . Therefore, using the additivity of the correspondence for functions of class  $C$ , we have

$$\Psi_1(U) + \Psi_5(U) = \Psi_4(U) + \Psi_2(U).$$

Hence

$$\Psi_1(U) - \Psi_2(U) = \Psi_4(U) - \Psi_5(U).$$

The correspondence constructed is monotonic also for functions in  $C_1$ , i.e., the inequality

$$\Psi_1(e^{i\varphi}) \geq \Psi_2(e^{i\varphi})$$

implies

$$\Psi_1(U) \geq \Psi_2(U).$$

Indeed, from the definition of the functions of the class  $C_1$  we have

$$\Psi_1(e^{i\varphi}) = \Psi_3(e^{i\varphi}) - \Psi_4(e^{i\varphi}),$$

$$\Psi_2(e^{i\varphi}) = \Psi_5(e^{i\varphi}) - \Psi_6(e^{i\varphi}),$$

where  $\Psi_i(e^{i\varphi}) \in C$ ,  $i = 3, 4, 5, 6$ .

From the condition

$$\Psi_3(e^{i\varphi}) - \Psi_4(e^{i\varphi}) \geq \Psi_5(e^{i\varphi}) - \Psi_6(e^{i\varphi})$$

we obtain

$$\Psi_3(e^{i\varphi}) + \Psi_6(e^{i\varphi}) \geq \Psi_5(e^{i\varphi}) + \Psi_4(e^{i\varphi}).$$

Hence by the monotonicity of the correspondence for functions of class  $C$  we have

$$\Psi_3(U) + \Psi_6(U) \geq \Psi_5(U) + \Psi_4(U),$$

i.e.,

$$\Psi_3(U) - \Psi_4(U) \geq \Psi_5(U) - \Psi_6(U),$$

i.e.,  $\Psi_1(U) \geq \Psi_2(U)$ .

In particular the class  $C_1$  contains the functions  $e_\mu(\varphi)$  defined for  $0 \leq \mu \leq 2\pi$  as follows:  $e_0(\varphi) \equiv 0$ ,  $e_{2\pi}(\varphi) \equiv 1$ , and for  $0 < \mu < 2\pi$

$$e_\mu(\varphi) = \begin{cases} 1, & \text{if } 2k\pi < \varphi \leq 2k\pi + \mu; \\ 0, & \text{if } 2k\pi + \mu < \varphi \leq 2(k+1)\pi, \quad k = 0, \pm 1, \dots \end{cases}$$

It is easy to see that the function  $e_0^1(\varphi)$  equal to 1 at the points  $\varphi = 2k\pi$  and zero at all other points belongs to the class  $C$ . We further introduce the function  $e_\mu^1(\varphi)$  for  $0 < \mu < 2\pi$ :

$$e_\mu^1(\varphi) = \begin{cases} 1, & \text{when } 2k\pi \leq \varphi \leq 2k\pi + \mu; \\ 0, & \text{when } 2k\pi + \mu < \varphi < 2(k+1)\pi, \end{cases}$$

which also belongs to  $C$ . By the fact that

$$e_\mu(\varphi) = e_\mu^1(\varphi) - e_0^1(\varphi),$$

we conclude that  $e_\mu(\varphi)$  belongs to the class  $C_1$  and we can assign to this function the operator

$$E(\mu) = e_\mu(U) \quad (E(0) = 0, \quad E(2\pi) = E).$$

Since the functions  $e_\mu(\varphi)$  coincide with their squares, it follows that they correspond to projections:  $E(\mu) = (E(\mu))^2$ ,  $E^*(\mu) = E(\mu)$ . If  $0 \leq r < \mu \leq 2\pi$  we have  $e_r(\varphi) \leq e_\mu(\varphi)$ , and therefore  $E(r) \leq E(\mu)$ . We shall now show that  $E(\mu)$  is continuous from the right as a function of  $\mu$ . The function  $e_\mu^1(\varphi)$  belongs to the class  $C$ , and consequently by Lemma 3 we can construct a decreasing sequence of trigonometric polynomials  $P_n(e^{i\varphi})$  that tends to  $e_\mu^1(\varphi)$ , and in such a way that for sufficiently large  $n$  the inequalities

$$P_n(e^{i\varphi}) \geq e_{\mu + \frac{1}{n}}^1(\varphi)$$

hold. Then for the corresponding operators we have, as  $n \rightarrow \infty$ ;

$$E^1\left(\mu + \frac{1}{n}\right) \rightarrow E^1(\mu).$$



By the representation

$$E^1(\mu) = E(\mu) + E^1(0),$$

which follows from the representation for the functions  $e_\mu^1(\varphi) = e_\mu(\varphi) + e_0^1(\varphi)$ , we obtain

$$E\left(\mu + \frac{1}{n}\right) \rightarrow E(\mu) \quad \text{as } n \rightarrow \infty,$$

i.e.,

$$\lim_{\lambda \rightarrow \mu+0} E(\lambda) = E(\mu).$$

Consider a partition of the closed interval  $[0, 2\pi]$  by points

$$0 = \mu_0 < \mu_1 < \cdots < \mu_n = 2\pi.$$

In each of the intervals  $[\mu_{k-1}, \mu_k]$  we choose an arbitrary point  $\varphi_k$ . For any fixed integer  $r$  and any  $\varphi \in [0, 2\pi]$  we have the inequality

$$\left| e^{ir\varphi} - \sum_{k=1}^n e^{ir\varphi_k} [e_{\mu_k}(\varphi) - e_{\mu_{k-1}}(\varphi)] \right| \leq |r| \max_k (\mu_k - \mu_{k-1}).$$

Indeed for  $\mu_{l-1} < \varphi \leq \mu_l$  we have

$$\begin{aligned} \left| e^{ir\varphi} - \sum_{k=1}^n e^{ir\varphi_k} [e_{\mu_k}(\varphi) - e_{\mu_{k-1}}(\varphi)] \right| &= |e^{ir\varphi} - e^{ir\varphi_l}| \\ &= 2 \left| \sin \frac{r(\varphi - \varphi_l)}{2} \right| \leq |r| |\varphi - \varphi_l| \leq |r| (\mu_l - \mu_{l-1}). \end{aligned}$$

The inequality also holds for  $\varphi = 0$ , since the left-hand side vanishes in that case. Hence we find that for a partition of the closed interval  $[0, 2\pi]$  of sufficiently small diameter

$$\overline{\left( e^{ir\varphi} - \sum_{k=1}^n e^{ir\varphi_k} [e_{\mu_k}(\varphi) - e_{\mu_{k-1}}(\varphi)] \right) \left( e^{ir\varphi} - \sum_{k=1}^n e^{ir\varphi_k} [e_{\mu_k}(\varphi) - e_{\mu_{k-1}}(\varphi)] \right)} \leq \varepsilon^2.$$

Passing to operators, we find

$$\begin{aligned} 0 &\leq \left( U^r - \sum_{k=1}^n e^{ir\varphi_k} [E(\mu_k) - E(\mu_{k-1})] \right)^* \left( U^r - \sum_{k=1}^n e^{ir\varphi_k} [E(\mu_k) - E(\mu_{k-1})] \right) \\ &\leq \varepsilon^2 E, \end{aligned}$$

from which it follows that

$$\left\| U^r - \sum_{k=1}^n e^{ir\varphi_k} [E(\mu_k) - E(\mu_{k-1})] \right\| \leq \varepsilon.$$

This last inequality proves the representation

$$U^r = \int_0^{2\pi} e^{ir\varphi} dE(\varphi).$$

The uniqueness of the family of projections  $E(\varphi)$  corresponding to the operator  $U$  follows from properties of the Stieltjes integral and is proved in analogy with the proof of uniqueness for the spectral family of a bounded symmetric operator (Theorem 11).

Let us summarize the results we have obtained:

**THEOREM 12.** *To every unitary operator  $U$  one can assign a unique family of projections  $E(\varphi)$  depending on the real parameter  $\varphi$ ,  $\varphi \in [0, 2\pi]$ , and having the properties*

- a)  $E(\varphi_1) \leq E(\varphi_2)$  for  $\varphi_1 \leq \varphi_2$ ;
- b)  $E(\varphi + 0) = E(\varphi)$ ;
- c)  $E(0) = 0$ ,  $E(2\pi) = E$ .

Using the family  $E(\varphi)$  the operator  $U$  can be represented in the form

$$U = \int_0^{2\pi} e^{i\varphi} dE(\varphi).$$

For any trigonometric polynomial and even for any continuous function  $u(e^{i\varphi})$  the relation

$$u(U) = \int_0^{2\pi} u(e^{i\varphi}) dE(\varphi)$$

holds, where the integral is understood as the limit of the corresponding approximating sums in the norm of the space.

### 2.13. Unbounded Operators

Many important operators are unbounded. Consider, for example the operator

$$T = -d^2/dx^2$$

in the Hilbert space  $H = L^2[0, \pi]$ . Let the domain of definition of the operator  $T$  consist of the infinitely differentiable functions on the closed interval  $[0, \pi]$  satisfying the conditions

$$y(0) = y(\pi) = 0.$$

Then the functions

$$y_n(x) = \sin nx$$

belong to the domain of definition and

$$Ty_n = n^2 y_n, \quad Ty_n = \lambda_n y_n, \quad \lambda_n = n^2.$$

Since the operator has arbitrarily large eigenvalues, it is not bounded. Thus an operator in a Hilbert space  $H$  is a linear mapping of some linear manifold  $D_T$  of the space  $H$  into the space  $H$ . Therefore in order to define an unbounded operator one must first describe its domain of definition, and then show how it acts on this domain. In the study unbounded operators the graph of an operator plays an important role.

DEFINITION 5. The *graph* of the operator  $T$  is the set of pairs

$$\{f, Tf\}$$

where  $f$  ranges over the domain  $D_T$ .

Consequently  $G_T$  is a subset of the Hilbert space

$$\mathcal{H} = H \oplus H,$$

consisting of all pairs  $\{f, g\}$  with  $f \in H, g \in H$  and with the basic operations introduced by the formulas

$$\begin{aligned} c\{f, g\} &= \{cf, cg\}, \\ \{f_1, g_1\} + \{f_2, g_2\} &= \{f_1 + f_2, g_1 + g_2\}, \\ (\{f_1, g_1\}, \{f_2, g_2\}) &= (f_1, f_2) + (g_1, g_2). \end{aligned}$$

Two operators  $T_1$  and  $T_2$  are called *coincident* if  $G_{T_1} = G_{T_2}$ . If  $G_{T_2} \supset G_{T_1}$ , the operator  $T_2$  is called an *extension* of the operator  $T_1$ . In this case we write  $T_2 \supset T_1$ . In other words  $T_2 \supset T_1$  if and only if  $D_{T_2} \supset D_{T_1}$  and  $T_2 f = T_1 f$  for all  $f \in D_{T_1}$ . The operator  $T$  is called *closed* if its graph is a closed subspace of the Hilbert space  $\mathcal{H} = H \oplus H$ . The operator  $T$  *admits a closure* if it has a closed extension.

DEFINITION 6. An operator  $T_1$  is called the *closure* of the operator  $T$  if  $G_{T_1} = \overline{G_T}$  (where  $\overline{G_T}$  is the closure of the linear manifold  $G_T$ ).

In this case the operator  $T_1$  is denoted  $\overline{T}$ . It is clear that the operator  $\overline{T}$  is the smallest closed extension of the operator  $T$ . It is natural to try to construct a closed extension of an operator  $T$  by taking the closure of



its graph. It turns out, however, that the subspace  $\overline{G_T}$  in general is not necessarily the graph of any operator. As an example consider the operator  $S$  on the Hilbert space  $l^2$  acting as follows:

$$Se_n = ne_1, \quad e_n = (\underbrace{0, 0, \dots, 0}_n, 1, 0, \dots), \quad n = 1, 2, \dots,$$

whose domain of definition consists of the set of finite sequences. The points of the form  $(e_n/n, e_1)$  belong to  $G_S$ . Consequently the point  $(0, e_1)$  belongs to  $\overline{G_S}$ . Therefore  $\overline{G_S}$  is not the graph of any operator, since under a linear transformation the zero vector must map to the zero vector.

Let  $T$  be a linear operator whose domain of definition  $D_T$  is everywhere dense in  $H$ . Let  $g$  be some element of  $H$  to which one can assign a certain element  $g^*$  in such a way that for all  $f \in D_T$  the equality

$$(Tf, g) = (f, g^*)$$

holds. The set of pairs  $g$  and  $g^*$  for which equality holds for any  $f \in D_T$  is nonempty since, as one can easily see, equality holds in any case for  $g = g^* = 0$ . Further, if  $D_T$  is dense in  $H$ , then the element  $g^*$  is uniquely determined by the element  $g$ . In fact, assuming the contrary, we have

$$(Tf, g) = (f, g_1^*), \quad (Tf, g) = (f, g_2^*) \quad \text{for all } f \in D_T.$$

From this we find that for any  $f \in D_T$

$$(f, g_1^*) - (f, g_2^*) = (f, g_1^* - g_2^*) = 0.$$

Since  $D_T$  is dense in  $H$ , the continuity of the inner product implies that  $g_1^* = g_2^*$ . Thus the correspondence

$$g^* = T^*g,$$

under which the equality  $(Tf, g) = (f, T^*g)$  holds for all  $f \in D_T$  defines a certain operator  $T^*$  called the *adjoint operator* to  $T$ . We shall show that  $T^*$  is a linear operator. Let  $g_1$  and  $g_2$  belong to the domain of definition of  $T^*$ . Then for all  $f \in D_T$  we have

$$\begin{aligned} (Tf, \alpha g_1 + \beta g_2) &= \bar{\alpha}(Tf, g_1) + \bar{\beta}(Tf, g_2) = \bar{\alpha}(f, T^*g_1) \\ &+ \bar{\beta}(f, T^*g_2) = (f, \alpha T^*g_1) + (f, \beta T^*g_2) = (f, \alpha T^*g_1 + \beta T^*g_2). \end{aligned}$$

Therefore  $\alpha g_1 + \beta g_2$  belongs to  $D_{T^*}$  and  $T^*(\alpha g_1 + \beta g_2) = \alpha T^*g_1 + \beta T^*g_2$ . We note the following relations, which follow immediately from the definition of the adjoint operator.

- a)  $(cT)^* = \bar{c} \cdot T^*$ ,  $c \neq 0$ .  
 b) If  $S \subset T$ , then  $S^* \supset T^*$ .  
 c)  $(T_1 + T_2)^* \supset T_1^* + T_2^*$  if the domain of definition of the operator  $T_1 + T_2$  is dense in  $H$ .

In the Hilbert space  $\mathcal{H} = H \oplus H$  we introduce a unitary operator  $U$  (the analog of a rotation through an angle of  $\pi/2$  radians in the plane):

$$U\{f, g\} = \{-g, f\}.$$

Let the domain of definition  $D_T$  of the operator  $T$  be dense in  $H$ . Then to define the adjoint operator  $T^*g = g^*$  we have the equation

$$(Tf, g) = (f, g^*),$$

which holds for all  $f \in D_T$ .

Using the operator  $U$  we can write this equation in the form

$$(U\{f, Tf\}, \{g, g^*\}) = 0.$$

This means that the elements of the space  $\mathcal{H}$  belonging to  $G_{T^*}$  are orthogonal to  $UG_T$ . We denote by  $M^\perp$  the set of vectors in the Hilbert space  $\mathcal{H}$  that are orthogonal to  $M \subset \mathcal{H}$ . Thus we have

$$G_{T^*} = (UG_T)^\perp.$$

Since the orthogonal complement forms a closed linear manifold, it follows from this relation that the operator  $T^*$  is closed.

For the ensuing work we shall have need of the following lemma.

LEMMA 4. *Let  $M$  be a subspace of the Hilbert space  $H$ . Then*

$$\overline{M} = (M^\perp)^\perp,$$

where  $\overline{M}$  is the closure of  $M$ .

PROOF: We denote by  $V$  the closure of the linear manifold  $M$ . Then  $V$  is a subspace of the Hilbert space  $H$ . We shall show that

$$V^\perp = M^\perp.$$

Indeed, if  $x \in M^\perp$ , then  $(x, y) = 0$  for any  $y \in M$ . For any element  $f \in V$  there exists a sequence  $\{y_n\}$  such that  $y_n \in M$ ,  $n = 1, 2, \dots$ , and  $y_n \rightarrow f$  as

$n \rightarrow \infty$ . By the continuity of the inner product we obtain  $(x, f) = 0$  for any  $f \in V$ , from which it follows that  $x \in V^\perp$ , and we have the inclusion

$$M^\perp \subset V^\perp.$$

The reverse inclusion  $V^\perp \subset M^\perp$  follows from the relation  $M \subset V$ . We shall now show that  $(V^\perp)^\perp = V$ , from which the assertion we wish to prove will follow. Indeed, by Theorem 2 of Sec. 4.1.4 we have the representation

$$H = V \oplus V^\perp$$

i.e., each vector  $h \in H$  has a unique representation in the form

$$h = x + y, \quad x \in V, y \in V^\perp.$$

If  $h \in V$ , then  $y = 0$  and for any vector  $v \in V^\perp$

$$(h, v) = (x, v) = 0,$$

from which we conclude that  $h \in (V^\perp)^\perp$ .

Conversely let

$$f \in (V^\perp)^\perp, \quad f = x + y, \quad x \in V, y \in V^\perp.$$

Then

$$0 = (f, y) = (y, y),$$

so that  $y = 0$  and consequently  $f \in V$ . ■

By Lemma 4 the equality  $G_{T^*} = (UG_T)^\perp$  can be written in the form

$$G_{T^*} = \mathcal{N} \ominus \overline{UG_T}.$$

Since  $U$  is unitary, we have  $\overline{UG_T} = U\overline{G_T}$ , and consequently

$$G_{T^*} = \mathcal{N} \ominus U\overline{G_T}.$$

A consequence of this relation is the following proposition.

**PROPOSITION 7.** *The following relation holds:*

$$Z(T^*) = H \ominus \overline{R(T)},$$

where  $Z(T^*)$  is the nullspace of the operator  $T^*$ ,  $R(T)$  is the range of values of the operator  $T$ , and  $\overline{R(T)}$  is the closure of  $R(T)$ .



PROOF: Let  $f \in Z(T^*)$ , so that

$$\{f, 0\} \in G_{T^*}.$$

By the equality  $G_{T^*} = \mathcal{H} \ominus U\overline{G}_T$  we have

$$\{f, 0\} \in \mathcal{H} \ominus U\overline{G}_T,$$

from which we find that

$$\{f, 0\} \perp \{T\varphi, -\varphi\} \quad \text{for any vector } \varphi \in D_T.$$

Thus

$$f \perp T\varphi \quad \text{for any } \varphi \in D_T,$$

i.e.,

$$f \perp \overline{R(T)}.$$

It is easy to see that the reasoning just carried out will work in the opposite order also. ■

**THEOREM 13.** *An operator  $T$  whose domain of definition is dense in  $H$  admits a closure if and only if  $D_{T^*}$  is dense in  $H$ . In this case*

$$\overline{T} = T^{**}.$$

PROOF: Let the domain of definition of the operator  $T^*$  be dense in  $H$ . Then from the relation  $G_{T^*} = \mathcal{H} \ominus U\overline{G}_T$  we have

$$G_{T^{**}} = \mathcal{H} \ominus UG_{T^*}.$$

Consider the closure of  $G_T$ , a linear manifold in  $\mathcal{H}$ . By Lemma 4 we have

$$\overline{G}_T = (G_T^\perp)^\perp.$$

Taking account of the relation  $U^2 = -E$ , we have

$$(G_T^\perp)^\perp = ((U^2 G_T)^\perp)^\perp.$$

We remark that since  $U$  is unitary

$$U(M^\perp) = (UM)^\perp$$

for any subspace  $M$  in  $\mathcal{H}$ . Consequently

$$((U^2 G_T)^\perp)^\perp = (U(UG_T)^\perp)^\perp = (UG_T)^\perp.$$

Thus

$$\overline{G_T} = (UG_T)^\perp.$$

Hence by the equality  $G_{T^{**}} = \mathcal{H} \ominus UG_T$ , we find that  $\overline{G_T}$  is the graph  $G_{T^{**}}$ , i.e.,  $\overline{T} = T^{**}$ . Conversely, assume that  $D_{T^*}$  is not dense in  $H$ . Then there exists a vector  $f \neq 0$  orthogonal to all the vectors in  $D_{T^*}$ . Consequently the element  $\{0, f\} \in \mathcal{H}$  will be orthogonal to all elements of the form  $\{-T^*g, g\}$  where  $g$  ranges over  $D_{T^*}$ . Hence the vector  $\{0, f\}$  will be orthogonal to  $U\{g, T^*g\}$  for  $g \in D_{T^*}$ . From this we find that  $(UG_T)^\perp$  is not the graph of an operator. By the equality  $\overline{G_T} = (UG_T)^\perp$  we find that  $\overline{G_T}$  is not the graph of an operator, and therefore the operator  $T$  does not admit a closure, contrary to hypothesis. ■

COROLLARY. If  $T$  admits a closure, then

$$T^* = (\overline{T})^* = T^{***} = (\overline{T})^*.$$

DEFINITION 7. A linear operator  $T$  in a Hilbert space  $H$  is called *symmetric* or *Hermitian* if its domain of definition  $D_T$  is dense in  $H$  and  $(Tf, g) = (f, Tg)$  for any  $f, g \in D_T$ . Equivalently: the operator  $T$  is symmetric if its domain of definition  $D_T$  is dense in  $H$  and

$$T \subset T^*.$$

We conclude from this that a symmetric operator always admits a closure. It follows from Theorem 13 that  $T^{**}$  is the smallest closed extension of  $T$ , and so we have the inclusions

$$T \subset T^{**} \subset T^*.$$

Since  $T^* = T^{***}$ , it follows from the last inclusion that the operator  $T^{**}$  is again symmetric. Therefore in studying symmetric operators below we shall always assume that they are closed.

DEFINITION 8. An operator  $T$  is *self-adjoint* if  $T = T^*$ .

THEOREM 14 (Criterion for self-adjointness). Let  $T$  be a symmetric operator in the Hilbert space  $H$ . Then the operator  $T$  is self-adjoint if and only if either of the following two conditions holds:

a)  $T$  is closed and  $Z(T^* \pm iE) = \{0\}$ ;

b)  $R(T \pm iE) = H$ .

PROOF: We shall prove that self-adjointness of  $T$  implies condition a). Let  $\varphi \in D_{T^*}$  and  $T^*\varphi = i\varphi$ . Then  $T\varphi = i\varphi$  and

$$(\varphi, T^*\varphi) = (T\varphi, \varphi) = i(\varphi, \varphi).$$

In addition

$$(\varphi, T^*\varphi) = (\varphi, i\varphi) = -i(\varphi, \varphi).$$

Consequently  $\varphi = 0$ , i.e.,

$$Z(T^* - iE) = \{0\}.$$

The proof that

$$Z(T^* + iE) = \{0\}$$

is similar. We shall now prove that condition a) implies condition b). By Proposition 7 we find that  $R(T \mp iE)$  is a dense subset in  $H$ . We shall show that  $R(T \mp iE)$  is closed. By the symmetry of  $T$ , for all  $\varphi \in D_T$  we have

$$\begin{aligned} \|(T \pm iE)\varphi\|^2 &= (T\varphi \pm i\varphi, T\varphi \pm i\varphi) \\ &= \|T\varphi\|^2 + (T\varphi, \pm i\varphi) + (\pm i\varphi, T\varphi) + (\pm i\varphi, \pm i\varphi) = \|T\varphi\|^2 + \|\varphi\|^2. \end{aligned}$$

From this relation it follows that if  $\varphi_n \in D_T$  and  $(T \pm iE)\varphi_n \rightarrow f_0$ , then  $\varphi_n$  converges to some vector  $\varphi_0$  and  $T\varphi_n$  also converges. Since  $T$  is closed, we have  $\varphi_0 \in D_T$  and  $(T \pm iE)\varphi_0 = f_0$ . Hence  $R(T \pm iE)$  is a closed set, and so

$$R(T \pm iE) = H.$$

We now show that condition b) implies that  $T$  is self-adjoint.

Let  $\varphi \in D_{T^*}$ . Since  $R(T - iE) = H$ , there exists  $f \in D_T$  such that  $(T - iE)f = (T^* - iE)\varphi$ . Since  $D_T \subset D_{T^*}$ , we have  $\varphi - f \in D_{T^*}$  and

$$(T^* - iE)(f - \varphi) = 0.$$

By condition b) we have  $R(T + iE) = H$  and so by Proposition 7

$$Z(T^* - iE) = \{0\}.$$

It follows from this that  $f = \varphi \in D_T$ . Then  $D_{T^*} = D_T$ , i.e.,  $T$  is self-adjoint. ■



We shall now prove the following simple criterion for self-adjointness of a symmetric operator.

**PROPOSITION 8.** *If for a symmetric operator  $T$  there exists a number  $\lambda$  such that the elements of the form  $(T - \lambda E)x$  and the elements of the form  $(T - \bar{\lambda}E)x$  ( $x \in D_T$ ) range over all of  $H$  as  $x$  ranges over  $D_T$ , then  $T$  is self-adjoint.*

**PROOF:** Let  $y \in D_{T^*}$ . Then for all  $x \in D_T$  we have

$$((T - \bar{\lambda}E)x, y) = (x, (T^* - \lambda E)y).$$

By the hypothesis of the theorem there exists  $h \in D_T$  such that

$$(T - \lambda E)h = (T^* - \lambda E)y.$$

Then by the symmetry of  $T$  we obtain

$$(x, (T^* - \lambda E)y) = (x, (T - \lambda E)h) = ((T - \bar{\lambda}E)x, h),$$

so that for all  $x \in D_T$

$$((T - \bar{\lambda}E)x, y) = ((T - \bar{\lambda}E)x, h).$$

Since  $(T - \bar{\lambda}E)x$  ranges over all of  $H$ , we have  $y = h \in D_T$ , so that  $T$  is self-adjoint. ■

#### EXAMPLE

In the Hilbert space  $L^2[0, 1]$  we introduce the operator

$$T = -\frac{d^2}{dx^2}$$

with domain of definition  $D_T$  consisting of the functions  $f(x)$  having the following properties:  $f(x)$  and  $f'(x)$  are both absolutely continuous on the closed interval  $[0, 1]$  and

$$\frac{d^2 f}{dx^2} \in L^2[0, 1], \quad f(0) = f(1) = 0.$$

**REMARK.** The function  $f(x)$  on  $[a, b]$  is called *absolutely continuous* if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\sum_{i=1}^n |f(x'_i) - f(x_i)| < \varepsilon$$

for any finite set of intervals  $[x_i, x'_i]$  satisfying the conditions  $x_i, x'_i \in [a, b]$  and

$$\sum_{i=1}^n |x'_i - x_i| < \delta.$$

For such functions we have the following theorem: *If  $f(x)$  is absolutely continuous on  $[a, b]$ , then  $f(x)$  is differentiable almost everywhere and  $f'(x) \in L^1[a, b]$ . Conversely, if  $g(x) \in L^1[a, b]$ , then the function*

$$G(x) = \int_a^x g(t) dt$$

*is absolutely continuous and  $G'(x) = g(x)$  almost everywhere. (Compare with the Radon-Nikodým theorem in Chapter 3).*

Integrating by parts twice, we verify that for  $f(x)$  and  $g(x)$  in  $D_T$

$$\begin{aligned} (Tf, g) = \int_0^1 -\frac{d^2 f(x)}{dx^2} \cdot \overline{g(x)} dx &= \left[ -\frac{df(x)}{dx} \cdot \overline{g(x)} \right. \\ &\quad \left. + \frac{dg(x)}{dx} \cdot f(x) \right] \Big|_0^1 + \int_0^1 f(x) \cdot \overline{\left( -\frac{d^2 g(x)}{dx^2} \right)} dx = (f, Tg). \end{aligned}$$

In addition the set  $D_T$ , as can easily be seen, is dense in  $L^2[0, 1]$ .

Thus  $T$  is a symmetric operator. We shall show that  $R(T) = L^2[0, 1]$ , from which it will follow by Proposition 8 that the operator  $T$  is self-adjoint. Indeed, for any function  $h(x) \in L^2[0, 1]$  we introduce the function  $f(x)$  by the formula

$$f(x) = -\int_0^x \left[ \int_0^t h(\tau) d\tau \right] dt + x \int_0^1 \left[ \int_0^t h(\tau) d\tau \right] dt.$$

By the remark above  $f(x) \in D_T$  and

$$Tf = h.$$

Since we wish to reduce the study of an unbounded symmetric operator to the study of a linear isometry, we introduce the so-called Cayley transform. Let  $T$  be a closed symmetric operator. The operator  $U = (T - iE) \cdot (T + iE)^{-1}$  is called the *Cayley transform* of the operator  $T$ . The existence of the operator  $(T + iE)^{-1}$  follows from the relations

$$\|(T \pm iE)h\|^2 = (Th, Th) \pm i(h, Th) \mp i(Th, h) + (h, h) = \|Th\|^2 + \|h\|^2,$$

which shows that the equation

$$(T + iE)h = 0$$

has only the zero solution. The domain of definition and the range of values of the operator  $U$  are respectively the sets of vectors of the forms

$$f = (T + iE)h, \quad g = (T - iE)h,$$

as  $h$  ranges over all of  $D_T$ .

We shall now prove that  $D_U$  and  $R(U)$  are closed sets, hence subspaces of  $H$ .

Let  $f_n = (T \pm iE)h_n$  and  $f_n \rightarrow f$  as  $n \rightarrow \infty$ . We have

$$\|f_n - f_m\|^2 = \|(T \pm iE)(h_n - h_m)\|^2 = \|T(h_n - h_m)\|^2 + \|h_n - h_m\|^2,$$

so that, since the sequence  $\{f_n\}$  is fundamental, it follows that

$$\lim_{n \rightarrow \infty} h_n \quad \text{and} \quad \lim_{m \rightarrow \infty} Th_m$$

both exist. Let  $\lim_{n \rightarrow \infty} h_n = h$ . Then since the operator  $T$  is closed, we conclude that

$$\lim_{n \rightarrow \infty} Th_n = Th.$$

Thus

$$f = (T \pm iE)h,$$

which proves that  $D_U$  and  $R(U)$  are closed. The operator  $U$  is therefore an isometry. Indeed, it follows from the relations

$$\|(T \pm iE)h\|^2 = \|Th\|^2 + \|h\|^2$$

that

$$\|(T - iE)h\| = \|(T + iE)h\|,$$

i.e., that

$$\|(T - iE)(T + iE)^{-1}f\| = \|f\|.$$

If we assume that the operator  $T$  is self-adjoint, then its Cayley transform is a unitary operator. To prove this assertion it suffices to prove that

$$D_U = R(U) = H,$$

and this follows from Theorem 14.



We shall show that the operator  $T$  can be unambiguously recovered from its Cayley transform. From the relations  $f = (T + iE)h$  and  $g = (T - iE)h$  we have

$$Uf = (T - iE)h.$$

Adding and subtracting the expression for  $Uf$  and  $f$ , we obtain

$$(E + U)f = 2Th, \quad (E - U)f = 2iEh.$$

It follows from this last equality that if  $(E - U)f = 0$ , then  $h = 0$ . But then by the relation  $f = (T + iE)h$ , we find that  $f = 0$ . Thus the operator  $(E - U)^{-1}$  exists and

$$T = i(E + U)(E - U)^{-1}.$$

We now turn to the proof of the spectral theorem for unbounded self-adjoint operators.

We shall prove the following lemma.

**LEMMA 4.** *Let  $H_1, H_2, \dots, H_i, \dots$  be a sequence of subspaces of the Hilbert space  $H$ , pairwise orthogonal and generating all of  $H$  when taken together. Denote the projection of an arbitrary element  $f$  on the subspace  $H_i$  by  $f_i$ . Let  $T_1, T_2, \dots, T_i, \dots$  be a sequence of linear operators with the property that on  $H_i$  the operator  $T_i$  behaves like a bounded self-adjoint operator mapping  $H_i$  into itself. Then there exists a unique self-adjoint operator  $T$  on  $H$  coinciding with  $T_i$  on each  $H_i$  ( $i = 1, 2, \dots$ ). The domain of definition of the operator  $T$  consists of the elements  $f$  for which the series*

$$\sum_{i=1}^{\infty} \|T_i f_i\|^2$$

*converges, and for such  $f$*

$$Tf = \sum_{i=1}^{\infty} T_i f_i. \quad (*)$$

**PROOF:** The operator  $T$  defined by the equality  $(*)$  is linear. The domain of definition  $D_T$  is dense in  $H$ , since it contains all elements of the form  $\sum_{i=1}^n f_i$ . The operator  $T$  is symmetric since for any  $f, g \in D_T$

$$(Tf, g) = \sum_{i=1}^{\infty} (T_i f_i, g_i) = \sum_{i=1}^{\infty} (f_i, T_i g_i) = (f, Tg).$$

Let  $g$  be any element of  $D_{T^*}$ . Then for any  $f \in D_T$

$$(Tf, g) = (f, T^*g),$$

from which it follows that

$$\sum_{i=1}^{\infty} (T_i f_i, g_i) = \sum_{i=1}^{\infty} (f_i, (T^*g)_i).$$

Take as  $f$  any element of the subspace  $H_j$ . Then the last equality assumes the form

$$(T_j f_j, g_j) = (f_j, (T^*g)_j).$$

By the hypothesis of the lemma  $T_j$  is self-adjoint on  $H_j$ , so that

$$(T_j f_j, g_j) = (f_j, T_j g_j).$$

Consequently

$$(T^*g)_j = T_j g_j,$$

from which we find

$$\sum_{j=1}^{\infty} \|T_j g_j\|^2 = \sum_{j=1}^{\infty} \|(T^*g)_j\|^2 = \|T^*g\|^2.$$

Thus  $g$  belongs to the domain of definition of the operator  $T$  and

$$Tg = \sum_{i=1}^{\infty} T_i g_i = \sum_{i=1}^{\infty} (T^*g)_i = T^*g.$$

It follows from this that  $T^* \subset T$ , and since  $T$  is symmetric, we have  $T^* = T$ .

To prove the uniqueness we assume that there exists another self-adjoint operator  $T'$  coinciding with  $T_i$  on  $H_i$ . By the fact that the operator  $T'$  is closed, its domain of definition will contain all  $f$  for which the series  $\sum_{i=1}^{\infty} T' f_i$  converges, and

$$\sum_{i=1}^{\infty} T' f_i = \sum_{i=1}^{\infty} T_i f_i = T' f.$$

The convergence of the series of orthogonal elements

$$\sum_{i=1}^{\infty} T_i f_i$$

is equivalent to the convergence of the series of their squared norms

$$\sum_{i=1}^{\infty} \|T_i f_i\|^2.$$

Therefore the set of such  $f$  coincides with  $D_T$ , and  $T'f = Tf$  for them. Thus  $T' \supset T$ . By the self-adjointness of the operators  $T$  and  $T'$  we have  $T' = T$ . ■

Let  $U = \int_0^{2\pi} e^{i\varphi} dF(\varphi)$  be the spectral decomposition of the unitary operator  $U$  that is the Cayley transform of the self-adjoint operator  $T$ . Using the correspondence

$$\lambda = -\cot \frac{\varphi}{2}, \quad 0 < \varphi < 2\pi,$$

between the interval  $(0, 2\pi)$  and the real line  $(-\infty, +\infty)$ , we obtain a family of projections

$$E(\lambda) = F(\varphi) = F(-2 \operatorname{arccot} \lambda), \quad -\infty < \lambda < +\infty.$$

Indeed, the function  $F(\varphi)$  is continuous at the point  $2\pi$  since  $(E - U)^{-1}$  exists and consequently 1 is not an eigenvalue of the operator  $U$ . Therefore we have

$$E(+\infty) = F(2\pi - 0) = E, \quad E(-\infty) = F(+0) = 0.$$

The properties  $E(\lambda) \leq E(\mu)$  for  $\lambda < \mu$  and  $E(\lambda + 0) = E(\lambda)$  obviously hold because of the monotonicity and continuity of the function  $\lambda = -\cot \frac{\varphi}{2}$ .

Consider the set of points  $\{\varphi_m\}$  on the interval  $(0, 2\pi)$  satisfying the equation

$$-\cot \frac{\varphi_m}{2} = m \quad (m = 0, \pm 1, \pm 2, \pm 3, \dots).$$

The projections

$$P_m = F(\varphi_m) - F(\varphi_{m-1})$$

are pairwise orthogonal, and in addition

$$\sum_{m=-\infty}^{+\infty} P_m = \lim_{k \rightarrow +\infty} F(\varphi_k) - \lim_{l \rightarrow -\infty} F(\varphi_l) = E - 0 = E.$$

We denote by  $H_m$  the closed subspace corresponding to the projection  $P_m$ . Since the operator  $P_m$  commutes with  $U$  and consequently with  $T = i(E +$



$U)(E - U)^{-1}$ , the subspace  $H_m$  reduces\* the operators  $U$  and  $T$ . The function  $(1 - e^{i\varphi})^{-1}$  is continuous in the interval  $\varphi_{m-1} \leq \varphi \leq \varphi_m$ , so that for  $f \in H_m$  we have

$$\begin{aligned} Tf &= TP_m f = i(E + U) \cdot (E - U)^{-1} \cdot P_m f \\ &= \int_{\varphi_{m-1}}^{\varphi_m} i(1 + e^{i\varphi}) \cdot (1 - e^{i\varphi})^{-1} dF(\varphi) f \\ &= \int_{\varphi_{m-1}}^{\varphi_m} \left( -\cot \frac{\varphi}{2} \right) dF(\varphi) f = \int_{m-1}^m \lambda dE(\lambda) f. \end{aligned}$$

We have now obtained the result that in each  $H_m$  the operator  $T$  acts like the bounded self-adjoint operator

$$T_m = \int_{m-1}^m \lambda dE(\lambda).$$

Since the subspaces  $H_0, H_{-1}, H_1, \dots$  are pairwise orthogonal and generate all of  $H$ , the operator  $T$  can be represented in the form

$$Tf = \sum_{-\infty}^{+\infty} \int_{m-1}^m \lambda dE(\lambda) f_m = \int_{-\infty}^{+\infty} \lambda dE(\lambda) f.$$

Thus we have proved

**THEOREM 15.** *To any self-adjoint operator  $G$  in the Hilbert space  $H$  one can associate a unique family of projections  $E(\lambda)$  depending on the real parameter  $\lambda$  with the following properties:*

- a)  $E(\lambda) \leq E(\mu)$  for  $\lambda \leq \mu$ ;
- b)  $E(\lambda + 0) = E(\lambda)$ ;
- c)  $E(\lambda) \rightarrow 0$  as  $\lambda \rightarrow -\infty$ ,  $E(\lambda) \rightarrow E$  as  $\lambda \rightarrow +\infty$ . Using  $E(\lambda)$  the operator is representable in the form

$$T = \int_{-\infty}^{+\infty} \lambda dE(\lambda).$$

---

\*If  $P$  is a projection that commutes with an operator  $T$  and  $Q = E - P$ ,  $H_P = PH$ ,  $H_Q = QH$ , the closed subspace  $H_P$  is said to *reduce* the operator  $T$  if  $T$  can be recovered from its parts  $T_P$  and  $T_Q$  acting in the subspaces  $H_P$  and  $H_Q$ , and the domain of definition of the operator  $T$  consists of the vectors whose projections on  $H_P$  and  $H_Q$  belong respectively to the domains of definition of the operators  $T_P$  and  $T_Q$ .

### 2.14. The Spectrum of a Symmetric Bounded Operator

In this section we shall study the properties of the spectrum of a symmetric operator. Much of what is said in this section is valid also for an unbounded operator, but we shall consider only bounded symmetric operators.

**THEOREM 16.** *A necessary and sufficient condition for the point  $\lambda$  to be a regular value of a bounded symmetric operator  $A$  is that there exist a positive constant  $r$  such that for any  $f \in H$*

$$\|Af - \lambda f\| \geq r\|f\|.$$

**PROOF.** **NECESSITY:** Let  $\lambda$  be a regular value. Then the operator  $R_\lambda = (A - \lambda E)^{-1}$  exists and  $\|R_\lambda\| = d < \infty$ ; for any vector  $f \in H$  we have

$$\|f\| = \|R_\lambda(A - \lambda E)f\| \leq d\|(A - \lambda E)f\|.$$

Therefore we have the relation

$$\|(A - \lambda E)f\| \geq \frac{1}{d}\|f\| = r\|f\|.$$

**SUFFICIENCY:** Let  $g = Af - \lambda f$  for any  $f \in H$ . Then  $g$  ranges over some linear manifold  $V$ . Since  $\|(A - \lambda E)f\| \geq r\|f\|$ , the correspondence between the vectors  $\{f\}$  and  $\{g\}$  is one-to-one; for if  $f_1$  and  $f_2$  go to the same vector  $g$ , then

$$Af_1 - \lambda f_1 - Af_2 + \lambda f_2 = (A - \lambda E)(f_1 - f_2).$$

Hence

$$\|f_1 - f_2\| \leq \frac{1}{r}\|(A - \lambda E)(f_1 - f_2)\| = 0, \quad \text{i.e., } f_1 = f_2.$$

We shall show that  $V$  is dense in  $H$  and closed, so that  $V = H$ ; then we can appeal to the bounded inverse theorem (cf. Chapter 2).

Suppose  $V$  is not dense in  $H$ . Then there exists a vector  $f_0 \neq 0$ ,  $f_0 \in H$  and such that  $(f_0, g) = 0$  for any vector  $g \in V$ . This means that  $(f_0, Af - \lambda f) = 0$  for any  $f \in H$ . Then  $Af_0 - \bar{\lambda}f_0 = 0$ ,  $f_0 \neq 0$ . But this is impossible: if  $\lambda$  is a complex number, the symmetric operator would have a

complex eigenvalue; if  $\lambda$  is real, then  $\lambda = \bar{\lambda}$  and  $\|f_0\| \leq \frac{1}{r} \|Af_0 - \lambda f_0\| = 0$ , and therefore  $f_0 = 0$ .

We shall now show that  $V$  is closed. Let  $\{g_n\} \subset V$  and  $g_n = (A - \lambda E)f_n$  as  $g_n \rightarrow g$ . Then

$$\|f_n - f_m\| \leq \frac{1}{r} \|(A - \lambda E)(f_n - f_m)\| = \frac{1}{r} \|g_n - g_m\|.$$

Since the sequence  $\{g_n\}$  converges, the sequence  $\{f_n\}$  also converges. Let  $f = \lim_{n \rightarrow \infty} f_n$ . And

$$(A - \lambda E)f = \lim_{n \rightarrow \infty} (A - \lambda E)f_n = \lim_{n \rightarrow \infty} g_n = g,$$

i.e.,  $g \in V$ . Thus  $V = H$ . In addition the correspondence  $g = (A - \lambda E)f$  is one-to-one. Therefore there exists a bounded inverse operator  $f = (A - \lambda E)^{-1}g = R_\lambda g$  defined on the whole space  $H$ . We have

$$\|(A - \lambda E)^{-1}g\| = \|f\| \leq \frac{1}{r} \|(A - \lambda E)f\| = \frac{1}{r} \|g\|, \quad \text{i.e., } \|R_\lambda\| \leq \frac{1}{r}.$$

Consequently  $\lambda$  is a regular value of the operator  $A$ . ■

**COROLLARY.** *The point  $\lambda$  belongs to the spectrum of the bounded symmetric operator  $A$  if and only if there exists a sequence  $\{f_n\}$  such that  $\|Af_n - \lambda f_n\| \leq c_n \|f_n\|$ , where  $c_n \rightarrow 0$  as  $n \rightarrow \infty$ . We remark that we may assume  $\|f_n\| = 1$  and then  $\|Af_n - \lambda f_n\| \rightarrow 0$ .*

**THEOREM 17.** *Complex numbers  $\lambda = \alpha + i\beta$  with  $\beta \neq 0$  are regular values of the bounded symmetric (self-adjoint) operator  $A$ .*

**PROOF:** If  $g = (A - \lambda E)f$ , then

$$(g, f) = (Af, f) - \lambda(f, f), \quad (f, g) = \overline{(g, f)} = (Af, f) - \bar{\lambda}(f, f).$$

Therefore

$$\begin{aligned} (f, g) - (g, f) &= (\lambda - \bar{\lambda})(f, f) = 2i\beta \|f\|^2, \\ 2|\beta| \cdot \|f\|^2 &\leq |(f, g)| + |(g, f)| \leq 2\|g\| \cdot \|f\|, \\ \|g\| &\geq |\beta| \cdot \|f\|, \quad \text{i.e., } \|(A - \lambda E)f\| \geq |\beta| \cdot \|f\|, \end{aligned}$$

and one now need only apply the preceding theorem. ■

**THEOREM 18.** *The spectrum of a symmetric (self-adjoint) operator  $A$  is entirely contained in the interval  $[m, M]$  of the real line, where  $m$  and*



$m$  and  $M$  are respectively the greatest lower bound and least upper bound of the operator.

PROOF: It follows from the preceding theorem that the spectrum of the operator  $A$  is contained in the real line. We shall prove that the points lying outside the interval  $[m, M]$  are regular points of the operator. For example, suppose  $\lambda > M$ ,  $\lambda = M + d$ ,  $d > 0$ . We have

$$((A - \lambda E)f, f) = (Af, f) - \lambda(f, f) \leq M(f, f) - \lambda(f, f) = -d\|f\|^2.$$

Hence

$$|((A - \lambda E)f, f)| \geq d\|f\|^2.$$

On the other hand we have the inequality

$$|((A - \lambda E)f, f)| \leq \|(A - \lambda E)f\| \cdot \|f\|.$$

Therefore  $\|(A - \lambda E)f\| \geq d\|f\|$ , which was to be proved. The proof of the case  $\lambda < m$  is similar. ■

THEOREM 19. The numbers  $m$  and  $M$  are points of the spectrum of the operator.

PROOF: We remark that if the operator  $A$  is replaced by the operator  $A_\mu = A - \mu E$ , its spectrum is shifted leftward by the amount  $\mu > 0$  and the numbers  $m$  and  $M$  are replaced by  $m - \mu$  and  $M - \mu$ . We shall first assume that  $0 \leq m \leq M$ . In such a case, as has been proved, we have the equality  $M = \|A\|$ . In studying completely continuous operators we proved that for a symmetric operator there exists a sequence  $\{f_n\}_{n=1}^\infty$  such that

$$\|f_n\| = 1, \quad n = 1, 2, 3, \dots, \quad \|Af_n - \|A\| \cdot f_n\| \rightarrow 0 \quad (n \rightarrow \infty).$$

(No use was made of complete continuity in the proof of this point.) To prove the theorem it now suffices to apply the corollary given after Theorem 16. ■

COROLLARY. Every bounded self-adjoint operator has a nonempty spectrum.

#### EXAMPLES

1. Let  $A = E$ . Then the spectrum of  $A$  consists of the single eigenvalue 1, corresponding to the eigenspace  $H_1 = H$ . For  $\lambda \neq 1$  the operator  $R_\lambda = \frac{1}{1 - \lambda}E$  is a bounded operator.

2. Let  $A$  be defined on  $L^2[0, 1]$  by the formula  $Af(x) = xf(x)$ . Obviously  $m = 0$ ,  $M \leq 1$ . We shall show that all points of  $[0, 1]$  belong to the spectrum of the operator  $A$  and therefore  $M = 1$ . Let  $0 \leq \lambda \leq 1$ . Consider\* the interval  $[\lambda, \lambda + \varepsilon] \subset [0, 1]$ . Let

$$f_\varepsilon(x) = \begin{cases} \frac{1}{\sqrt{\varepsilon}}, & \text{for } x \in [\lambda, \lambda + \varepsilon], \\ 0 & \text{for } x \notin [\lambda, \lambda + \varepsilon]. \end{cases}$$

We have

$$\|f_\varepsilon(x)\|^2 = \int_0^1 f_\varepsilon^2(x) dx = 1, \quad \text{i.e., } \|f_\varepsilon(x)\| = 1.$$

Further

$$(A - \lambda E)f_\varepsilon(x) = (x - \lambda)f_\varepsilon(x),$$

$$\|(A - \lambda E)f_\varepsilon(x)\|^2 = \frac{1}{\varepsilon} \int_\lambda^{\lambda+\varepsilon} (x - \lambda)^2 dx = \frac{\varepsilon^2}{3}.$$

As  $\varepsilon \rightarrow 0$  we have  $\|(A - \lambda E)f_\varepsilon\| \rightarrow 0$ , so that  $\lambda$  is a point of the spectrum for any  $\lambda$  satisfying the inequalities  $0 \leq \lambda \leq 1$ . At the same time the operator  $A$  has no eigenvalues (this is proved just as in the space  $C[0, 1]$ ). Thus the operator has only a continuous spectrum.

**THEOREM 20.** *A necessary and sufficient condition for  $\lambda_0$  to be an eigenvalue of the symmetric (self-adjoint) operator  $A$  is that  $\lambda_0$  be a point of discontinuity of the function  $E(\lambda)$ .*

**PROOF. NECESSITY:** Suppose for some  $f_0 \neq 0$

$$Af_0 - \lambda_0 f_0 = 0.$$

Then  $((A - \lambda_0 E)^2 f_0, f_0) = 0$  and consequently

$$\int_{m-0}^M (\lambda - \lambda_0)^2 d(E(\lambda)f_0, f_0) = 0.$$

Since the integrand is nonnegative and the integrator is monotonically increasing, we have also  $\int_\alpha^\beta (\lambda - \lambda_0)^2 d(E(\lambda)f_0, f_0) = 0$  for any half-open

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\*For  $\lambda = 1$  it is necessary to consider the interval  $[1 - \varepsilon, 1]$ .

interval  $(\alpha, \beta]$ . In particular  $\int_{\lambda_0+\varepsilon}^M (\lambda - \lambda_0)^2 d(E(\lambda)f_0, f_0) = 0$  for any  $\varepsilon > 0$ . Since  $(\lambda - \lambda_0)^2 \geq \varepsilon^2$ , we have

$$\varepsilon^2 \int_{\lambda_0+\varepsilon}^M d(E(\lambda)f_0, f_0) = \varepsilon^2 [(f_0, f_0) - (E(\lambda_0 + \varepsilon)f_0, f_0)] = 0.$$

Consequently

$$(f_0, f_0) - (E(\lambda_0 + \varepsilon)f_0, f_0) = 0, \quad E(\lambda_0 + \varepsilon)f_0 = f_0.$$

Similarly  $E(\lambda_0 - \varepsilon)f_0 = 0$ . Therefore  $E(\lambda_0 + \varepsilon)f_0 - E(\lambda_0 - \varepsilon)f_0 = f_0$ . Therefore  $(E(\lambda_0 + \varepsilon) - E(\lambda_0 - \varepsilon))f_0 = f_0 \neq 0$ , i.e.,  $\lambda_0$  is indeed a point of discontinuity for  $E(\lambda)$ , and the eigenvector  $f_0$  belongs to the subspace corresponding to the operator

$$E(\Delta) = E(\lambda_0) - E(\lambda_0 - \varepsilon).$$

SUFFICIENCY: Let  $E(\lambda_0 - 0) \neq E(\lambda_0)$  and let  $f_0$  be any element of the subspace corresponding to the operator  $E(\Delta) = E(\lambda_0) - E(\lambda_0 - \varepsilon)$ . Then  $(E(\lambda_0) - E(\lambda_0 - \varepsilon))f_0 = f_0$ , i.e.,  $f_0$  belongs to the orthogonal complement of the space  $G_{\lambda_0-0}$  in the space  $G_{\lambda_0}$ .\* Therefore  $E(\lambda_0)f_0 = f_0$ ,  $E(\lambda_0 - 0)f_0 = 0$ , and a fortiori  $E(\lambda)f_0 = 0$  for  $\lambda < \lambda_0$ . Consequently  $E(\Delta)f_0 = f_0$  for  $\Delta = (\lambda_0 - \varepsilon, \lambda_0]$ . But then

$$\begin{aligned} Af_0 &= AE(\Delta)f_0 = \int_{\lambda_0-\varepsilon}^{\lambda_0} \lambda dE(\lambda)f_0, \\ \lambda_0 f_0 &= \lambda_0 E(\Delta)f_0 = \int_{\lambda_0-\varepsilon}^{\lambda_0} \lambda_0 dE(\lambda)f_0. \end{aligned}$$

Consequently

$$Af_0 - \lambda_0 f_0 = \int_{\lambda_0-\varepsilon}^{\lambda_0} (\lambda - \lambda_0) dE(\lambda)f_0.$$

Hence

$$\|Af_0 - \lambda_0 f_0\| \leq \varepsilon \|E(\Delta)f_0\| \leq \varepsilon \|f_0\|.$$

Since  $\varepsilon$  is arbitrary, we have  $\|Af_0 - \lambda_0 f_0\| = 0$ . It has simultaneously been proved that the entire subspace onto which the operator  $E(\Delta) = E(\lambda_0) -$

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\*Here  $G_{\lambda_0-0}$  and  $G_{\lambda_0}$  are the subspaces on which the operators  $E(\lambda_0 - 0)$  and  $E(\lambda_0)$  respectively project the whole space.



$E(\lambda_0 - 0)$  projects consists of eigenvectors of the operator  $A$  corresponding to the eigenvalue  $\lambda_0$ . ■

## 2.15. The Spectrum and Resolvent of Unbounded Operators

Just as in the case of bounded operators (cf. Sec. 4.2.6) one can introduce the concepts of resolvent set and spectrum for unbounded operators also.

Just as for bounded operators, if for some  $\lambda_1$  the equation  $Tf = \lambda_1 f$  has a nonzero solution  $f$ , the number  $\lambda_1$  is called an eigenvalue of the operator  $T$  and the solution  $f$  is called an eigenvector of corresponding to the eigenvalue  $\lambda_1$ .

The set of all eigenvectors  $\{f\}$  corresponding to  $\lambda_1$  together with the zero vector is called the eigenspace  $H_{\lambda}$  corresponding to the eigenvalue  $\lambda_1$ . Its dimension is called the multiplicity of the eigenvalue  $\lambda_1$ .

Let  $D_T$  be the domain of definition and  $R(T)$  the range of values of the unbounded linear operator  $T$  on the Hilbert space  $H$ . Let  $B(\lambda) = T - \lambda E$ . The numbers  $\lambda$  for which the range of values of the operator  $B(\lambda)$  is dense in  $H$  and there exists a continuous inverse  $B^{-1}(\lambda) = (T - \lambda E)^{-1}$  are called the *regular values* of the operator  $T$  (the values that belong to the resolvent set). The operator  $B^{-1}(\lambda) = (T - \lambda E)^{-1}$  is called the *resolvent operator* or the *resolvent* of the operator  $T$  and denoted  $R_{\lambda}$ . Thus

$$R_{\lambda} = (T - \lambda E)^{-1} = B^{-1}(\lambda).$$

The set complementary to the resolvent (in the complex plane) is called the *spectrum* of the operator  $T$  and denoted  $\sigma(T)$ .

The following classification of the spectrum of an operator can be given.

1. The set of complex numbers  $\lambda$  for which  $B(\lambda)$  does not have an inverse is called the *point spectrum*. It obviously coincides with the set of eigenvalues of the operator.

2. The set of complex numbers  $\lambda$  for which the operator  $B(\lambda)$  possesses an inverse with a dense domain, but  $B^{-1}(\lambda)$  is not continuous is called the *continuous spectrum*.

3. The set of complex numbers for which the operator  $B(\lambda)$  has an inverse whose domain of definition is not dense in  $H$  is called the *residual spectrum*.

Let us consider some examples.

### EXAMPLES

1. Let  $H = L^2(-\infty, \infty)$  and let the operator  $T$  be multiplication by the independent variable:

$$Tf = xf.$$

Let us describe the domain of definition  $D_T$  of this operator. It is obvious that the functions  $f(x)$  and  $xf(x)$  must both belong to  $L^2(-\infty, \infty)$ . On such functions  $Tf = xf$ . Let  $Tf = xf = \lambda f$ . We shall show that then any real number  $\lambda$  belongs to the continuous spectrum of the operator  $T$ . The operator has no eigenvalues, since if  $(x - \lambda)f(x) = 0$  almost everywhere, then  $f(x) = 0$  almost everywhere, i.e.,  $f = 0$  as an element of  $L^2(-\infty, \infty)$ . Thus the homogeneous equation has only the trivial solution and the operator  $R_\lambda = B^{-1}(\lambda) = (T - \lambda E)^{-1}$  exists.

We remark that the set of functions  $\{g(x)\}$  in  $L^2(0, 1)$  that vanish on some neighborhood of the point  $\lambda$  (a different neighborhood of  $\lambda$  for each function) is contained in the domain of definition of the operator  $R_\lambda$ . Consequently the domain of  $R_\lambda$  is dense in  $L^2(-\infty, \infty)$ . The operator  $R_\lambda = \frac{1}{x - \lambda}$  is unbounded on this domain. Thus, every number  $\lambda \in \mathbb{R}$  belongs to the continuous spectrum.

2. Consider the operator  $T = i \frac{d}{dx}$  mapping  $L^2(0, 1)$  into itself. Let the domain of definition of this operator consist of the absolutely continuous functions  $\varphi(x)$  on  $[0, 1]$  having  $\varphi'(x) \in L^2(0, 1)$  and satisfying the condition

$$\varphi(0) = \varphi(1) = 0.$$

It is obvious that  $D_T$  is dense in  $L^2(0, 1)$  and the operator  $T$  is unbounded. In addition the operator  $T$  is symmetric: for any  $\varphi, \psi \in D_T$  we have

$$(T\varphi, \psi) = \int_0^1 i\varphi' \bar{\psi} dx = i\varphi \bar{\psi} \Big|_0^1 + \int_0^1 \varphi \overline{(i\psi')} dx = \int_0^1 \varphi \overline{T\psi} dx = (\varphi, T\psi).$$

We remark that the relation

$$(T\varphi, \psi) = (\varphi, T\psi)$$

will hold also in the case when  $\varphi \in D_T$  and the function  $\psi(x)$  is absolutely continuous and  $\psi'(x) \in L^2(0, 1)$ . Therefore  $\psi \in D_{T^*}$  and

$$T^* \psi = i \frac{d}{dx} \psi.$$

It turns out that the set of absolutely continuous functions  $\psi$  having square summable derivatives is the domain of definition of the operator  $T^*$  and

$$T^* \psi = i\psi'.$$

Let  $\psi \in D_{T^*}$ . Then for any  $\varphi \in D_T$  we have:

$$\begin{aligned}
 (T\varphi, \psi) &= (\varphi, T^*\psi) = (\varphi, \psi^*) \\
 &= \int_0^1 \varphi(x) \overline{\psi^*(x)} dx = -i \int_0^1 \varphi(x) \frac{d}{dx} \left\{ \int_0^x i\psi^*(t) dt + C \right\} dx \\
 &= -i \varphi(x) \left\{ - \int_0^x i\psi^*(t) dt + C \right\} \Big|_0^1 + i \int_0^1 \varphi'(x) \overline{\left\{ - \int_0^x i\psi^*(t) dt + C \right\}} dx \\
 &= \int_0^1 i\varphi'(x) \overline{\left\{ - \int_0^x i\psi^*(t) dt + C \right\}} dx,
 \end{aligned}$$

where  $C$  is an arbitrary constant.

We have now found that for any  $\varphi \in D_T$

$$\int_0^1 \varphi'(x) \overline{\left\{ \psi + \int_0^x i\psi^*(t) dt - C \right\}} dx = 0.$$

We choose the constant  $C$  from the following requirement:

$$\int_0^1 \left\{ \psi(x) + \int_0^x i\psi^*(t) dt - C \right\} dx = 0.$$

Then the function

$$\varphi_0(x) = \int_0^x \left\{ \psi(s) + \int_0^s i\psi^*(t) dt - C \right\} ds$$

belongs to the domain of definition of the operator  $T$  and the relation

$$\int_0^1 \varphi'(x) \overline{\left\{ \psi(x) + \int_0^x i\psi^*(t) dt - C \right\}} dx = 0$$

can be rewritten for  $\varphi = \varphi_0$  in the form

$$\int_0^1 \left| \psi(x) + \int_0^x i\psi^*(t) dt - C \right|^2 dx = 0.$$

Consequently, almost everywhere

$$\psi(x) = -i \int_0^x \psi^*(t) dt + C,$$

i.e.,

$$i\psi'(x) = \psi^*(x) = T^*\psi.$$



It follows from this that  $\psi(x)$  is absolutely continuous on  $[0, 1]$  and  $\psi'(x) \in L^2[0, 1]$ .

We shall show, for example, that the points  $\lambda = \pm i$  belong to the residual spectrum of this operator.

Indeed, for any function  $\varphi(x) \in D_T$  and  $e^x \in D_T$  we have

$$((T + iE)\varphi, e^x) = (\varphi, (T^* - iE)e^x) = (\varphi, ie^x - ie^x) = 0.$$

The element  $e^x$  is orthogonal to the linear manifold  $R(T + iE)$  and consequently  $R(T + iE)$  is not dense in  $L^2[0, 1]$ . It can be shown similarly that the element  $e^{-x}$  is orthogonal to  $R(T - iE)$ , i.e.,  $\lambda = -i$  also belongs to the residual spectrum of the operator  $T$ .

Let  $T$  be a self-adjoint operator in  $H$ . Then the following theorem holds.

**THEOREM 21.** *Every complex number  $\lambda$  for which  $\text{Im } \lambda \neq 0$  belongs to the resolvent set of a self-adjoint operator  $T$ . For such  $\lambda$  the resolvent  $R_\lambda$  is a bounded operator satisfying*

$$\|R_\lambda\| \leq \frac{1}{|\text{Im } \lambda|}.$$

Moreover

$$\text{Im}((T - \lambda E)f, f) = -\text{Im } \lambda \|f\|^2 \quad f \in D_T.$$

**PROOF:** For  $f \in D_T$  we have  $(Tf, f) = (f, Tf) = \overline{(Tf, f)}$ , i.e., the inner product is real-valued. From this we find that

$$\text{Im}((T - \lambda E)f, f) = \text{Im}(Tf, f) - \text{Im } \lambda (f, f) = -\text{Im } \lambda \|f\|^2.$$

By the Cauchy-Bunyakovskii inequality we have

$$\|(T - \lambda E)f\| \|f\| \geq |(T - \lambda E)f, f| = |(Tf, f) - \lambda(f, f)| \geq |\text{Im } \lambda| \|f\|^2,$$

i.e.,

$$\|(T - \lambda E)f\| \geq |\text{Im } \lambda| \|f\|.$$

Consequently the inverse operator  $(T - \lambda E)^{-1}$  exists. We shall show that the range of values of the operator  $(T - \lambda E)$  is dense in  $H$  if  $\text{Im } \lambda \neq 0$ . Suppose, to the contrary, there exists a vector  $h \perp R(T - \lambda E)$ , where  $R(T - \lambda E)$  is the range of values of the operator  $(T - \lambda E)$  and  $h \neq 0$ . Then

$$0 = ((T - \lambda E)f, h) = (f, (T - \bar{\lambda}E)h), \quad f \in D_T.$$

But  $D_T$  is dense in  $H$  and so  $(T - \bar{\lambda}E)h = 0$ , i.e.,  $Th = \bar{\lambda}h$ , contradicting the fact that  $(Th, h)$  is real-valued. Consequently for  $\text{Im } \lambda \neq 0$  the resolvent  $R_\lambda$  exists and is bounded and  $\|R_\lambda\| \leq \frac{1}{|\text{Im } \lambda|}$ . ■

**THEOREM 22.** *Let  $T$  be a closed linear operator on  $H$ . Then for any  $\lambda$  in the resolvent set the resolvent  $R_\lambda = (T - \lambda E)^{-1}$  is a continuous (bounded) operator defined on all of  $H$ .*

**PROOF:** Since  $\lambda$  belongs to the resolvent set,  $D_{(T - \lambda E)^{-1}} = R(T - \lambda E)$  is dense in  $H$ , and there exists a constant  $d > 0$  such that

$$\|(T - \lambda E)f\| \geq d\|f\|, \quad f \in D_T.$$

We shall show that the range of values of the operator  $T - \lambda E$  coincides with all of  $H$ .

Suppose that for some sequence  $\{f_n\}$  the sequence  $(T - \lambda E)f_n$  has a limit in  $H$  equal to  $g$ , i.e.

$$\lim_{n \rightarrow \infty} (T - \lambda E)f_n = g.$$

Then the limit  $\lim_{n \rightarrow \infty} f_n = f$  also exists. But the operator  $T$  is closed, and so  $(T - \lambda E)f = g$ . Consequently  $R(T - \lambda E) = H$ , since by hypothesis  $R(T - \lambda E) = H$ . ■

#### EXERCISES

1. Construct an example of an operator on a Hilbert space  $H$  whose range of values is not closed.

2. Let  $H$  be a Hilbert space having an orthonormal basis  $\{\varphi_j\}$ . Let the linear operator  $A$  be defined by the rule

$$A\varphi_j = \lambda_j \varphi_j, \quad \lambda_j \in \mathbb{C}.$$

Prove that the spectrum of such an operator coincides with the closure of the set  $\{\lambda_j\}$ . Show that every closed bounded subset of the complex plane is the spectrum of some operator of this type.

3. In the space  $l^p$  the operator  $S$  acts on any vector  $\xi \in l^p$ ,  $\xi = (\xi_1, \xi_2, \dots)$  as follows:  $S\xi = \xi_0 = (\xi_2, \xi_3, \dots)$ . Find the spectrum of the operator  $S$  for  $p \geq 1$ .

4. Consider the operator  $B$  on the space  $L^2[0, 1]$ :

$$Bf(x) = \int_0^x f(y) dy, \quad f \in L^2[0, 1].$$

Does the operator  $B$  map any nonzero vector into zero? Does the operator  $B^*B$  have a finite absolute norm? Prove that  $B + B^*$  is a projection on the one-dimensional subspace consisting of the constants. Show that the spectrum of the operator  $A = (E + B)^{-1}$  consists of the point  $\{1\}$  and that  $\|A\| = 1$ .

5. The spread of two linear manifolds of  $H$  is the norm of the difference of the projections on their closures. Thus if the spread of two subspaces  $M_1$  and  $M_2$  is denoted  $\theta(M_1, M_2)$ , then  $\theta(M_1, M_2) = \|P_2 - P_1\|$ , where  $P_1$  and  $P_2$  are the projections on  $\overline{M_1}$  and  $\overline{M_2}$  respectively. Prove that if  $\theta(M_1, M_2) < 1$ , then  $M_1$  and  $M_2$  are of the same dimension.

6. Let the operator  $A$ , bounded and defined on the whole space  $H$ , be such that every element of the form  $Af$  can be represented in the form:

$$Af = \sum_{i=1}^{\infty} \mu_i (f, \varphi_i) \varphi_i,$$

where  $\{\varphi_i\}_1^{\infty}$  is an orthonormal sequence and  $\mu_i \rightarrow 0$  as  $i \rightarrow \infty$ . Prove that then  $A$  is completely continuous.

7. For a bounded linear operator  $A$  the limit  $\lim_{n \rightarrow \infty} \|A^n\|^{1/n} = r(A)$  exists (cf. Theorem 1 in Sec. 4.3.1 below). It is called the *spectral radius* of the operator  $A$ . Prove that  $r(\alpha A) = |\alpha| r(A)$  for any number  $\alpha$ , and if the operators  $A$  and  $B$  commute (i.e.,  $AB = BA$ ) then

$$r(A + B) \leq r(A) + r(B), \quad r(AB) \leq r(A)r(B).$$

8. Find the spectrum and spectral radius of the operators  $A$  on  $L^2[0, 1]$  given by the following formulas:

$$Ax(t) = \int_0^1 \ln(ts)x(s) ds, \quad Ax(t) = \int_0^t \left(\frac{s}{t}\right)^{\alpha} x(s) ds,$$

and for the operators  $A : C[0, 1] \rightarrow C[0, 1]$  given by the formulas

$$Ax(t) = \int_0^t x(s) ds, \quad Ax(t) = tx(t).$$

9. Show that the spectral radius of the operator  $A : C[0, 1] \rightarrow C[0, 1]$  defined by the formula  $Ax(t) = \int_0^t K(t, s)x(s) ds$ , where  $K(t, s)$  is a continuous function of two variables on the square  $0 \leq s, t \leq 1$ , is equal to zero.



10. Show that the spectrum of the operator  $A : C[0, 1] \rightarrow C[0, 1]$  defined by the formula  $Ax(t) = x(t^2)$  is contained in the unit disk.

11. Find the eigenvectors and eigenvalues of the integral operator on  $L^2[0, 1]$  given by the formula

$$Ax(t) = \int_0^1 K(t, s)x(s) \, ds,$$

where  $K(t, s) = \cos 2\pi(t - s)$  or  $K(t, s) = \min(t, s)$ .

12. In the Hilbert space  $H = L^2(-\infty, \infty)$  consider the operator of multiplication by the independent variable  $Tf(x) = xf(x)$  with domain of definition  $D_T$  consisting of functions for which

$$\int_{-\infty}^{+\infty} x^2 |f(x)|^2 \, dx < \infty.$$

Prove that  $T$  is a self-adjoint operator and its spectral decomposition has the form

$$Tf = \int_{-\infty}^{+\infty} \lambda \, dE(\lambda)f = \int_{-\infty}^{\infty} \lambda \, d(e_{\lambda}(x)f(x)),$$

where

$$e_{\lambda}(x) = \begin{cases} 1 & \text{if } x \leq \lambda, \\ 0, & \text{if } x > \lambda. \end{cases}$$

3. OPERATOR EQUATIONS. ANALYTIC FUNCTIONS AND OPERATORS

3.1. Analytic Properties of the Resolvent

Throughout the following the action takes place in a Hilbert space  $H$ . The operators that occur here are not in general self-adjoint. Such operators are called *nonself-adjoint*.

DEFINITION 1. A vector-valued (resp. operator-valued) function on the Hilbert space  $H$  is a function whose value is a vector  $h(\lambda)$  (resp. bounded linear operator  $A(\lambda)$ ) in  $H$  for each value of the parameter  $\lambda$  in the scalar field  $P$ .

DEFINITION 2. A vector-valued function  $h(\lambda)$  (resp. operator-valued function  $A(\lambda)$ ) in  $H$  is called an *analytic* function of the complex parameter  $\lambda$  in some region  $G$  of the  $\lambda$ -plane if at each point  $\lambda \in G$  the ratio  $\frac{h(\lambda + \Delta\lambda) - h(\lambda)}{\Delta\lambda}$  (resp.  $\frac{A(\lambda + \Delta\lambda) - A(\lambda)}{\Delta\lambda}$ ) converges in the norm of  $H$

(resp. the uniform norm) to some limit  $h'(\lambda)$  (resp.  $A'(\lambda)$ ) that is a vector-valued (resp. operator-valued) function.

Vector-valued and operator-valued functions have all the basic properties of scalar-valued analytic functions of a complex variable. In particular, for  $h(\lambda)$  and  $A(\lambda)$  there is a residue theorem and a Cauchy integral formula. In a neighborhood of an isolated singularity  $\lambda_0$  there exists a Laurent expansion

$$h(\lambda) = \sum_{-\infty}^{+\infty} \frac{h_n}{(\lambda - \lambda_0)^n}, \quad A(\lambda) = \sum_{-\infty}^{+\infty} \frac{A_n}{(\lambda - \lambda_0)^n},$$

that converges in norm and locally uniformly with respect to  $\lambda$ . Here  $h_n \in H$ , and  $A_n$  are bounded linear operators on  $H$ .

We remark that in the case of vector-valued (resp. operator-valued) functions a pole, an essential singularity, and a removable singularity are defined in analogy with the scalar-valued case. If the only singularities of the functions  $h(\lambda)$  and  $A(\lambda)$  in  $G$  are poles, i.e., only a finite number of terms of negative degree in  $(\lambda - \lambda_0)$  occur in their Laurent expansions, then  $h(\lambda)$  and  $A(\lambda)$  are called *meromorphic* functions.

A function  $h(\lambda)$  or  $A(\lambda)$  is called *entire* if it is analytic in the entire complex plane. The *order* of an entire function  $h(\lambda)$  (resp.  $A(\lambda)$ ) is the number

$$\rho_h = \overline{\lim}_{|\lambda| \rightarrow \infty} \frac{\ln \ln \|h(\lambda)\|}{\ln |\lambda|} \quad \left( \text{resp. } \rho_A = \overline{\lim}_{|\lambda| \rightarrow \infty} \frac{\ln \ln \|A(\lambda)\|}{\ln |\lambda|} \right).$$

The type of an entire function  $h(\lambda)$  or  $A(\lambda)$  is defined in analogy to the scalar case. The maximum principle and propositions similar to Lindelöf's theorem hold for the functions  $\|h(\lambda)\|$  or  $\|A(\lambda)\|$ .

Consider for example the case when  $A(\lambda)$  is independent of  $\lambda$  and is a "scalar" operator, i.e., let  $A$  be a bounded linear operator on the Hilbert space  $H$ . Let  $\lambda_0$  be a regular value of the bounded operator  $A$ , i.e., at the point  $\lambda_0$  there exists a bounded operator  $R_{\lambda_0} = (A - \lambda_0 E)^{-1}$  defined on the entire space  $H$ . Then, as is known from Sec. 4.2.6, there exists a neighborhood of the point  $\lambda_0$  such that all the points of this neighborhood are also regular points of the operator  $A$ . We shall show that the resolvent operator  $R_\lambda = (A - \lambda E)^{-1}$  defined on this neighborhood is an analytic operator-valued function, and that  $\left(\frac{d}{d\lambda}\right)^n R_\lambda = n! R_\lambda^{n+1}$ .

In fact, by Hilbert's identity for the resolvent

$$R_\lambda - R_z = (\lambda - z) R_\lambda R_z,$$

which is easily derived from the following relation

$$\begin{aligned} R_\lambda - R_z &= R_\lambda (A - zE) R_z - R_\lambda (A - \lambda E) R_z \\ &= R_\lambda A R_z - z R_\lambda R_z - R_\lambda A R_z + \lambda R_\lambda R_z = (\lambda - z) R_\lambda R_z, \end{aligned}$$

it follows that  $R_\lambda$  and  $R_z$  commute. In addition  $R_\lambda = [1 - (z - \lambda) R_\lambda] R_z$ . Expanding the expression in brackets in a series about  $\lambda = \lambda_0$ , we arrive at the relation (in which for convenience we have replaced  $z$  by  $\lambda$ )

$$R_\lambda = [1 - (\lambda - \lambda_0) R_{\lambda_0}]^{-1} R_{\lambda_0} = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}^{n+1}.$$

The series on the right-hand side converges absolutely at least for those  $\lambda$  satisfying the condition

$$|\lambda - \lambda_0| < \|R_{\lambda_0}\|^{-1}.$$

The expansion of  $R_\lambda$  (in a neighborhood of the point  $\lambda_0$ )

$$R_\lambda = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}^{n+1}$$

is called the *Neumann series* for the resolvent.

This expansion shows that  $R_\lambda$  is a holomorphic function of  $\lambda$  whose Taylor series has the form

$$\sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}^{n+1}.$$

Consequently

$$\left(\frac{d}{d\lambda}\right)^n R_\lambda = n! R_{\lambda_0}^{n+1}, \quad n = 1, 2, 3, \dots$$

For  $|\lambda| > \|A\|$  the resolvent  $R_\lambda$  admits the expansion (cf. Sec. 4.2.6)

$$R_\lambda = -\lambda^{-1} (1 - \lambda^{-1} A)^{-1} = -\sum_{n=0}^{\infty} \lambda^{-n-1} A^n.$$

It follows from these considerations that the spectrum of the operator  $A$  is always nonempty. Indeed, if such were not the case,  $R_\lambda$  would be an entire function that is bounded on the entire complex plane, hence by Liouville's



theorem for entire functions it would follow that  $R_\lambda = \text{const.}$  But since  $R_\lambda \rightarrow 0$  as  $\lambda \rightarrow \infty$ , we should have  $R_\lambda = 0$ .

We remark also that every eigenvalue  $\lambda_0$  of the operator  $A$  is a singular point of the analytic function  $R_\lambda$ . Indeed, suppose to the contrary that  $\lambda_0$  is a regular point (a removable singularity of the analytic function  $R_\lambda$ ). Then the following limit exists in the norm of the space

$$\lim_{\lambda \rightarrow \lambda_0} R_\lambda = R_{\lambda_0}.$$

Therefore

$$(A - \lambda_0 E)R_{\lambda_0} = \lim_{\lambda \rightarrow \lambda_0} (A - \lambda E)R_\lambda = E.$$

Thus the inverse  $(A - \lambda_0 E)^{-1} = R_{\lambda_0}$  exists, i.e.,  $\lambda_0$  is not an eigenvalue. We have now reached a contradiction.

**THEOREM 1.** *For a bounded linear operator  $A$  the following limit, called the spectral radius, exists:*

$$\lim_{n \rightarrow \infty} \|A^n\|^{1/n} = r(A).$$

*The estimate  $r(A) \leq \|A\|$  holds. If  $|\lambda| > r(A)$ , then the resolvent  $R_\lambda$  exists and can be represented as a norm-convergent series*

$$R_\lambda = - \sum_{n=0}^{\infty} \lambda^{-n-1} A^n.$$

**PROOF:** Indeed, let  $r_0 = \inf_{n \geq 1} \|A^n\|^{1/n}$ . Then if we prove the inequality  $\overline{\lim}_{n \rightarrow \infty} \|A^n\|^{1/n} \leq r_0$ , the existence of the limit  $\lim_{n \rightarrow \infty} \|A^n\|^{1/n}$  will also be demonstrated.

For each  $\varepsilon > 0$  we choose  $k$  so that  $\|A^k\|^{1/k} \leq r_0 + \varepsilon$ . For an integer  $n$  we determine  $q$  by the formula  $n = pk + q$ ,  $0 \leq q \leq k - 1$  ( $p$  is an integer). Since  $\|AB\| \leq \|A\| \cdot \|B\|$ , we have

$$\|A^n\|^{1/n} \leq \|A^k\|^{p/n} \cdot \|A\|^{q/n} \leq (r_0 + \varepsilon)^{kp/n} \|A\|^{q/n}.$$

Since  $pk/n \rightarrow 1$  and  $q/n \rightarrow 0$  as  $n \rightarrow \infty$ , it follows that

$$\overline{\lim}_{n \rightarrow \infty} \|A^n\|^{1/n} \leq r_0 + \varepsilon.$$

Since  $\varepsilon$  is arbitrary, we find that  $\overline{\lim}_{n \rightarrow \infty} \|A^n\|^{1/n} \leq r_0$ . The existence of the limit  $r(A) = \lim_{n \rightarrow \infty} \|A^n\|^{1/n}$  is now proved. Further,  $\|A^n\| \leq \|A\|^n$ , and so  $\lim_{n \rightarrow \infty} \|A^n\|^{1/n} \leq \|A\|$ , i.e.,  $r(A) \leq \|A\|$ . From this we find that the series for  $R_\lambda$  converges for  $|\lambda| > r(A)$ . Indeed, if  $|\lambda| \geq r(A) + \varepsilon$ ,  $\varepsilon > 0$ , we have  $\|\lambda^{-n} A^n\| = |\lambda|^{-n} \cdot \|A^n\| \leq (r(A) + \varepsilon)^{-n} (r(A) + \frac{\varepsilon}{2})^n$  for sufficiently large  $n$  (here we have used the inequality for  $|\lambda|$  and the property of the limit  $r(A) = \lim_{n \rightarrow \infty} \|A^n\|^{1/n}$ ). Therefore the series of  $R_\lambda$  with general term  $\lambda^{-n} A^n$  converges in norm. That this is the series for the resolvent can be verified by multiplying it on the left and right by  $(A - \lambda E)$ . ■

We remark that for a symmetric operator  $A$  we have the following equality for the spectral radius:

$$r(A) = \|A\|.$$

Indeed, since

$$(A^2 f, f) = (A f, A f) = \|A f\|^2,$$

we have the following equalities:  $\|A^2\| = \|A\|^2$ ,  $\|A^4\| = \|A\|^4$ , ...,  $\|A^{2^m}\| = \|A\|^{2^m}$ . For any arbitrary exponent  $n$  we choose  $m$  so that  $2^m - n = p \geq 0$ . Then  $\|A\|^{n+p} = \|A^{n+p}\| \leq \|A^n\| \cdot \|A^p\| \leq \|A^n\| \cdot \|A\|^p$ , i.e.,  $\|A\|^n \leq \|A^n\|$ . Since the opposite inequality is always true, we have

$$r(A) = \lim_{n \rightarrow \infty} (\|A^n\|)^{1/n} = \|A\|.$$

Therefore it is easy to see that for the norm of the resolvent  $R_\lambda$  of a symmetric operator  $A$  we have the relations

$$\|R_\lambda\| = r(R_\lambda) = \frac{1}{d(\lambda, S(A))},$$

where  $d(\lambda, S(A))$  is the distance from a point  $\lambda$  where the resolvent  $R_\lambda$  is defined to the spectrum  $S(A)$  of the operator  $A$ .

Since the region of convergence of the series for the resolvent is the set  $|\lambda| > r(A)$ , it follows that on the boundary of the circle of convergence, i.e., for  $|\lambda| = r(A)$ , there exists at least one point of the spectrum of the operator  $A$ . If the radius of convergence of the series is zero, then the point  $\lambda = 0$  is a point of the spectrum, since otherwise the resolvent would be an entire function that is bounded in the entire plane, which we know to be impossible.

Thus we can now summarize the results we have obtained.

The spectrum of a bounded linear operator  $A$  in a Hilbert (or Banach) space is contained in the disk  $|\lambda| \leq r(A)$ . On the boundary of the disk there

is at least one point of the spectrum. Outside this disk the resolvent  $R_\lambda$  is representable in the form of a series that converges in the operator norm

$$R_\lambda = - \sum_{n=1}^{\infty} \lambda^{-n-1} A^n.$$

For a bounded symmetric operator the following equalities hold

$$r(A) = \|A\|, \quad \|R_\lambda\| = r(R_\lambda) = d^{-1}(\lambda, S(A)).$$

We now continue our study of operator-valued functions.

We shall denote by  $A^*(\lambda)$  the operator-valued function whose value at each  $\lambda_0 \in G$  is the operator  $A^*(\lambda_0)$  adjoint to the operator  $A(\bar{\lambda}_0)$ .

In what follows we shall call an operator-valued function  $A(\lambda)$  defined in the region  $G$  simply an operator. In doing this, of course, it is understood that  $A(\lambda_0)$  is a linear operator for each  $\lambda_0 \in G$ .

**DEFINITION 3.** The operator  $A(\lambda)$  is *completely continuous in the region  $G$*  if the operator  $A(\lambda_0)$  is completely continuous for each point  $\lambda_0$  of the region  $G$ .

We remark that if the operator  $A(\lambda)$  is completely continuous in a neighborhood of an isolated singularity  $\lambda = \lambda_0$ , then the Laurent coefficients are completely continuous operators. This fact follows from the representation of  $A_n$  in the form of an integral

$$A_n = \frac{1}{2\pi i} \oint A(\lambda)(\lambda - \lambda_0)^{-n-1} d\lambda$$

and Theorem 2 of Sec. 4.2.2. Indeed, replacing the integral by a finite sum, we represent  $A_n$  as the limit of a norm-convergent sequence of completely-continuous operators.

We remark also that the integral written above is the integral of an operator-valued function. It is the limit in the uniform norm of the operator approximating sums

$$\sum_i A(\mu_i)(\mu_i - \lambda_0)^{-n-1} \Delta\lambda_i,$$

where  $\mu_i$  are the points of division and the limit is taken as the diameter of the partition tends to zero. In the present case we are taking partitions of a circle containing the singularity  $\lambda_0$  in its interior. Such an integral has all the basic properties of the integral of a scalar function.



We have the following lemma.

LEMMA 1. Let  $\lambda \in F \subset G$ , where  $F$  is a closed bounded subset of the complex plane. Let  $A(\lambda)$  be a completely continuous operator that is an analytic operator-valued function in the region  $G$ , and let the vector  $f$  belong to the bounded set  $M$  of the Hilbert space  $H$ . Then each infinite sequence from the set  $\{A(\lambda)f\}$ ,  $\lambda \in F$ ,  $f \in M$ , contains a convergent subsequence.

PROOF: The proof of the lemma is obtained from the complete continuity of the operator  $A$  together with the fact that every infinite set  $\{\lambda\}$  contained in the closed subset  $F$  of the complex plane contains a convergent sequence. ■

Let us continue our study.

Let  $\{\varphi_n\}$  be an orthonormal basis of  $H$ . We write a matrix representation of a completely continuous operator function  $A(\lambda)$  that is analytic in the region  $G$ . If  $g(\lambda) = A(\lambda)f$ , then  $g_k = g_k(\lambda) = (A(\lambda)f, \varphi_k)$  and  $g = (g_1, g_2, \dots)$ . Let

$$f = \sum_{i=1}^{\infty} f_i \varphi_i, \quad f^N = \sum_{i=1}^N f_i \varphi_i.$$

Then

$$Af^N = \sum_{i=1}^N f_i A\varphi_i = \sum_{i=1}^N f_i \sum_{k=1}^{\infty} a_{ki} \varphi_k,$$

where  $A\varphi_i = \sum_{k=1}^{\infty} a_{ki} \varphi_k$ . Passing to the limit as  $N \rightarrow \infty$ , we obtain

$$\lim_{N \rightarrow \infty} Af^N = Af = \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ki} f_i \right) \varphi_k.$$

But  $Af = \sum_{k=1}^{\infty} g_k \varphi_k$ . Thus  $g_k = \sum_{i=1}^{\infty} a_{ki} f_i = (Af, \varphi_k) = (g, \varphi_k)$ .

We shall now prove the following lemma.

LEMMA 2. If a completely continuous operator  $A(\lambda)$  is an analytic operator-valued function of  $\lambda$  in the region  $G$ , then for each closed bounded set  $F \subset G$  one can exhibit a sequence of numbers  $\varepsilon_n \rightarrow 0$  for which the following inequalities hold:

$$\begin{aligned} \sum_{k=n}^{\infty} \left| \sum_{i=1}^{\infty} a_{ki} f_i \right|^2 &< \varepsilon_n \sum_{i=1}^{\infty} |f_i|^2, \\ \sum_{i=n}^{\infty} \left| \sum_{k=1}^{\infty} a_{ki} f_i \right|^2 &< \varepsilon_n \sum_{i=1}^{\infty} |f_i|^2. \end{aligned}$$

PROOF: It suffices to establish only the first inequality, as the second is a consequence of the complete continuity of the operator  $A^*(\lambda)$ . Dividing both sides of the first inequality by  $\|f\|^2 = \sum_{i=1}^{\infty} |f_i|^2$ , we arrive at the inequality

$$\sum_{k=n}^{\infty} \left| \sum_{i=1}^{\infty} a_{ki} l_i \right|^2 < \varepsilon_n, \quad l_i = \frac{f_i}{\|f\|}.$$

Then  $\sum_{i=1}^{\infty} |l_i|^2 = 1$ , i.e.,  $\|l\| = 1$ , if  $l = (l_1, l_2, \dots)$ . By the preceding lemma any set of elements  $\{A(\lambda)l\}$ ,  $\lambda \in F$ ,  $\|l\| = 1$ , contains a sequence that converges in  $H$ . In other words the set  $\{A(\lambda)l\}$ ,  $\lambda \in F$ ,  $\|l\| = 1$ , is compact. Then if  $g^0(\lambda) = A(\lambda)l$  and  $g^0(\lambda) = (g_1^0(\lambda), g_2^0(\lambda), \dots, g_k^0(\lambda), \dots)$ , we have  $g_k^0(\lambda) = \sum_{i=1}^{\infty} a_{ki} l_i$ . Consequently the required inequalities can be written as follows:

$$\sum_{k=n}^{\infty} |g_k^0(\lambda)|^2 < \varepsilon_n, \quad \lambda \in F, \quad \varepsilon_n \rightarrow 0.$$

However, according to Exercise 9 of Sec. 1.3.3 these inequalities do hold. This is what was to be proved. ■

We remark that the converse is obviously also true. If the inequalities written in the hypotheses of Lemma 2 hold, then the set  $\{A(\lambda)l\}$  is compact. Thus the property of a completely continuous operator established in Lemma 1 is equivalent to the validity of the first inequality proved in Lemma 2.

DEFINITION 4. The *resolvent*  $R(\lambda_0)$  of the operator  $A(\lambda_0)$  is the operator for which the relation

$$(E + R(\lambda_0))(E - A(\lambda_0)) = E$$

holds, where  $E$  is the identity operator.

If the resolvent exists, then one can obviously write the relation

$$E + R(\lambda_0) - (E + R(\lambda_0))A(\lambda_0) = E,$$

i.e.,

$$R(\lambda_0) = (E + R(\lambda_0))A(\lambda_0) = A(\lambda_0) + R(\lambda_0)A(\lambda_0).$$

Hence under the assumption that  $A(\lambda_0)$  is a completely continuous operator and  $R(\lambda_0)$  is a bounded operator, we find, using the properties of

completely continuous operators, that  $R(\lambda_0)$  is also a completely continuous operator.

The following theorem is of great importance in the theory of nonself-adjoint operators.

**THEOREM 2.** *Let  $A(\lambda)$  be a completely continuous operator for any value  $\lambda$  belonging to some region  $G$ , and for  $\lambda \in G$  let  $A(\lambda)$  be an analytic operator-valued function. If for some  $\lambda = \lambda_0 \in G$  there exists a bounded resolvent operator  $R(\lambda_0)$ , then  $R(\lambda)$  exists in the entire region  $G$  except possibly at a set of isolated points and is a meromorphic function of  $\lambda$ .*

**PROOF:** Consider the equation

$$g = A(\lambda)g + f,$$

which in matrix notation has the following form:

$$g_k = \sum_{i=1}^{\infty} a_{ki} g_i + f_k, \quad k = 1, 2, \dots,$$

where

$$g = (g_1, g_2, \dots), \quad f = (f_1, f_2, \dots).$$

The subspace spanned by vectors of the form  $(g_1, g_2, \dots, g_n, 0, 0, \dots)$  will be denoted by  $H_n$ , and the subspace spanned by  $(0, \dots, 0, x_{n+1}, x_{n+2}, \dots)$  will be denoted by  $H_{n+1}$ . Let  $P$  and  $P_1$  be the projections corresponding to these subspaces. We denote by  $F_0$  an arbitrary closed bounded subregion of the region  $G$  containing the point  $\lambda_0$ . Let  $\varepsilon_1, \varepsilon_2, \dots$  be the numbers corresponding to the set  $F_0$  in the inequalities of Lemma 2. We choose  $n$  so large that  $\varepsilon_{n+1} < 1$ . We write the first  $n$  equations of the system

$$g_k = \sum_{i=1}^{\infty} a_{ki} g_i + f_k \text{ in the operator form}$$

$$Pg = PAPg + PAP_1g + Pf, \quad (*)$$

and the remaining equations as follows:

$$P_1g = P_1APg + P_1AP_1g + P_1f. \quad (**)$$

According to the choice of the index  $n$  we have

$$\|P_1A\| < \varepsilon_{n+1}, \quad \lambda \in F_0.$$



Therefore the operator  $E - P_1 A$  has a resolvent in the region  $F_0$ , since it has an inverse in that region. For the resolvent  $R_1(\lambda)$  in  $F_0$  we obviously have the expansion

$$E + R_1 = E + P_1 A + (P_1 A)^2 + \dots$$

By the uniform convergence of the series it is also obvious that the resolvent is an analytic function of  $\lambda$  and

$$\|E + R_1\| \leq 1 + \varepsilon_{n+1} + \varepsilon_{n+1}^2 + \dots = \frac{1}{1 - \varepsilon_{n+1}}.$$

We apply the operator  $E + R_1$  on the left to both sides of equality (\*\*). We have

$$(E + R_1)P_1 g = (E + R_1)P_1 A P g + (E + R_1)P_1 A P_1 g + (E + R_1)P_1 f.$$

Hence

$$P_1 g = (E + R_1)P_1 A P g + (E + R_1)P_1 f + [(E + R_1)P_1 A P_1 - R_1 P_1]g.$$

The operator in brackets above is zero, since

$$(E + R_1)(P_1 A - E + E)P_1 - R_1 P_1 = -P_1 + (E + R_1)P_1 - R_1 P_1 = 0.$$

Therefore the equation (\*\*) is equivalent to the equation

$$P_1 g = (E + R_1)P_1 A P g + (E + R_1)P_1 f. \quad (**)'$$

Similarly we conclude that the equation (\*) is equivalent to the equality

$$P g = P A [E + (E + R_1)P_1 A] P g + P [E + A(E + R_1)P_1] f. \quad (*)'$$

This last equation is a system of linear algebraic equations in  $g_1, g_2, \dots, g_n$  whose coefficients, according to what was said above, are holomorphic functions of the parameter  $\lambda \in F_0$ . Moreover the operator  $E - A(\lambda_0)$  has an inverse. Therefore the equation  $[E - A(\lambda)]g = f$  and, in particular, the equation (\*)' has a solution for any right-hand side  $f$ . Then, as is well-known from linear algebra, the determinant  $\Delta_n(\lambda)$  of the system (\*)' is nonzero at the point  $\lambda_0$  and the solution of the system (\*)' can be written in the form

$$P g = \frac{L(\lambda) P [E + A(E + R_1)P_1] f}{\Delta_n(\lambda)},$$

where  $L(\lambda)$  is a certain “solving” operator depending analytically on  $\lambda$ . An explicit form for it can be found, for example, by Cramer’s rule. The function  $\Delta_n(\lambda)$  is a holomorphic function on  $F_0$  and vanishes at a finite number of points. Substituting the expression found above for  $P_1g$  using the “solving” operator into the equation  $(**)'$ , we can write

$$P_1g = (E + R_1)P_1A \frac{L(\lambda)P[E + A(E + R_1)P_1]f}{\Delta_n(\lambda)} + (E + R_1)P_1f.$$

Finally we obtain

$$\begin{aligned} g = Pg + P_1g &= (E + R)f = \frac{L(\lambda)P[E + A(E + R_1)P_1]f}{\Delta_n(\lambda)} \\ &+ \frac{(E + R_1)P_1AL(\lambda)P[E + A(E + R_1)P_1]f}{\Delta_n(\lambda)} + (E + R_1)P_1f. \end{aligned}$$

Therefore

$$E + R = \frac{1}{\Delta_n(\lambda)}\{E + (E + R_1)P_1A\}\{L(\lambda)P[E + A(E + R_1)P_1]\} + (E + R_1)P_1.$$

All the operators occurring on the right-hand side are analytic functions in  $F_0$ . The only singularities of the right-hand side are the zeros of the function  $\Delta_n(\lambda)$ , which are the poles of the function  $E + R$ , and there is only a finite number of them. The theorem is now proved. ■

3.2. Keldysh’s Theorem

DEFINITION 5. Let  $A(\lambda)$  be a completely continuous operator depending analytically on the parameter  $\lambda \in G$ . A nontrivial solution  $g$  of the equation  $g = A(\lambda_0)g$  is called an *eigenvector* of the operator  $A(\lambda)$ . The corresponding value of the parameter  $\lambda_0$  is called an *eigenvalue* of the operator corresponding to the given eigenvector  $g$ .

DEFINITION 6. An element  $g_k$  is called an *conjugate vector of order  $k$*  to the eigenvector  $g$  if it satisfies the following conditions.:

$$\begin{aligned} g &= A(\lambda_0)g, \\ g_1 &= A(\lambda_0)g + \frac{1}{1!} \frac{\partial A(\lambda_0)}{\partial \lambda} g, \\ &\dots\dots\dots \\ g_k &= A(\lambda_0)g_k + \frac{1}{1!} \frac{\partial A(\lambda_0)}{\partial \lambda} g_{k-1} + \dots + \frac{1}{k!} \frac{\partial^k A(\lambda_0)}{\partial \lambda^k} g. \end{aligned}$$

The elements  $g, g_1, g_2, \dots$  are said to form a *chain of conjugate vectors*. We remark that in the case when  $A(\lambda)$  is a polynomial in  $\lambda$ ,

$$A(\lambda) = A_0 + \lambda A_1 + \dots + \lambda^m A_m,$$

a function  $u(t)$  of the form

$$u(t) = e^{\lambda t} \left( g_k + \frac{t}{1!} g_{k-1} + \cdots + \frac{t^k}{k!} g \right)$$

satisfies the equation

$$u = A_0 u + A_1 \frac{\partial u}{\partial t} + \cdots + A_m \frac{\partial^m u}{\partial t^m}.$$

Therefore the problems of justifying the application of the Fourier method, for example, of finding solutions of such operator equations lead to the proof of completeness theorems for eigenvectors and conjugate vectors of the operator  $A(\lambda)$ . It is known that for a completely continuous operator with a given nonzero eigenvalue the number of linearly independent eigenvectors is finite (cf. Sec. 4.2.10). It can be shown that in the case when the resolvent  $R(\lambda)$  of the operator  $A(\lambda)$  is a meromorphic function of  $\lambda$ , the order of the conjugate vectors does not exceed the order of the pole of the resolvent at the value  $\lambda = \lambda_0$ .

Let  $g$  be an eigenvector. We denote by  $m$  the maximal order of the conjugate vectors to  $g$ . The number  $m + 1$  is called the *multiplicity* of the eigenvector  $g$ .

**DEFINITION 7.** A canonical system of eigenvectors and conjugate vectors at  $\lambda = \lambda_0$  is a system

$$g^{(k)}, g_1^{(k)}, \dots, g_{m_k}^{(k)}, \quad k = 1, 2, \dots$$

having the following properties:

- a) the vector  $g^{(1)}$  is an eigenvector whose multiplicity attains a possible maximum of  $m_1 + 1$ ;
- b) the vector  $g^{(k)}$  is an eigenvector that is linearly independent of  $g^{(1)}, \dots, g^{(k-1)}$  and whose multiplicity attains a possible maximum of  $m_k + 1$ ;
- c) the vectors  $g^{(k)}, g_1^{(k)}, \dots, g_{m_k}^{(k)}$  form a chain of conjugate vectors;
- d) the vectors  $\{g^{(k)}\}$  form a basis of the subspace of eigenvectors for  $\lambda = \lambda_0$ .

The number  $N = m_1 + 1 + m_2 + 1 + \cdots$  is called the *multiplicity of the eigenvalue*  $\lambda = \lambda_0$ .

It follows directly from the definition of a canonical system of eigenvectors and conjugate vectors that an arbitrary conjugate vector of order  $p$  is a linear combination of the vectors  $g_i^{(k)}$  for  $i \leq p$ . The multiplicity of an



eigenvector  $c_1 g^{(1)} + \dots + c_\nu g^{(\nu)}$  for  $c_\nu \neq 0$  is  $m_\nu$ . The numbers  $m_1, m_2, \dots$  are independent of the choice of the canonical system.

DEFINITION 8. A system of eigenvectors and conjugate vectors of a completely continuous operator  $A(\lambda)$  is called *n-complete* if any set of  $n$  vectors  $f_0, f_1, \dots, f_{n-1}$  can be represented as the limit (in the norm of the space) of linear combinations

$$f_{\nu, N} = \sum_{k=1}^N \sum_{(p)} a_{p, N}^{(k)} g_p^{(k, \nu)}, \quad \nu = 0, 1, \dots, n-1,$$

with coefficients that are independent of  $\nu$ , where

$$g_p^{(k, \nu)} = \left[ \frac{d^\nu}{dt^\nu} e^{\lambda_k t} \left( g_p^{(k)} + g_{p-1}^{(k)} \frac{t}{1!} + \dots + g^{(k)} \frac{t^p}{p!} \right) \right]_{t=0}$$

and  $\lambda_k$  are the eigenvalues of the completely continuous operator  $A(\lambda)$ .

In particular for  $n = 1$  this definition coincides with the usual definition of completeness of a system of eigenvectors and conjugate vectors. In the case when the multiplicity of all eigenvectors is 1, the vector  $f_{\nu, N}$  has the form

$$f_{\nu, N} = \sum_{k=1}^N a_N^{(k)} \lambda_k^\nu g^{(k)}.$$

Let  $B$  be a completely continuous self-adjoint operator. Consider the eigenvalue problem

$$\varphi = \mu B \varphi.$$

We denote by  $\mu_i$  the eigenvalues of this problem and by  $\varphi_i$  an orthogonal system of eigenvectors. We agree that if  $B\varphi_i = 0$ , then  $\mu_i = \infty$ .

DEFINITION 9. The operator  $B$  is *complete* if the system of eigenvectors  $\varphi_i$  corresponding to the eigenvalues  $\mu_i \neq \infty$  is complete in  $H$ .

DEFINITION 10. The greatest lower bound of the numbers  $\rho'$  for which  $\sum_{i=1}^{\infty} |\mu_i|^{-\rho'} < \infty$  is called the *order*  $\rho$  of the operator  $B$ .

We denote by  $B^\alpha$ ,  $\alpha > 0$ , the operator  $B^\alpha = \int \lambda^\alpha dE(\lambda)$ , where  $E(\lambda)$  is the partition of the identity determined by the operator  $B$ .

The following completeness theorem plays an important role in the questions we have touched upon.

**THEOREM 3 (Keldysh).** *Let  $B$  be a complete self-adjoint completely continuous operator of finite order, and let  $A_0, A_1, \dots, A_{n-1}$  be arbitrary completely continuous operators. For each of the equations*

$$\begin{aligned} g &= (A_0 + \lambda B^{\frac{1}{n}} A_1 + \dots + \lambda^{n-1} B^{\frac{n-1}{n}} A_{n-1} + \lambda^n B)g, \\ g &= (A_0^* + \lambda A_1^* B^{\frac{1}{n}} + \dots + \lambda^{n-1} A_{n-1}^* B^{\frac{n-1}{n}} + \lambda^n B)g \end{aligned}$$

*the system of eigenvectors and conjugate vectors is  $n$ -fold complete.*

### 3.3. Root Vectors and Root Subspaces of Nonself-Adjoint Operators

We now continue our study of the spectral properties of bounded linear operators defined on all of a Hilbert space  $H$ . Just as in the preceding section the operators we consider are in general nonself-adjoint.

**DEFINITION 11.** A vector  $f \neq 0$  is called a *root vector* for the eigenvalue  $\lambda_0$  of the linear operator  $A$  if there exists a natural number  $s$  such that

$$(A - \lambda_0 E)^s f = 0.$$

The set of all root vectors of the operator  $A$  corresponding to a given eigenvalue  $\lambda_0$ , together with the zero vector, forms a linear manifold  $L_{\lambda_0}$  called a *root manifold*. The dimension of this linear manifold is called the *algebraic multiplicity* of the eigenvalue  $\lambda_0$ . If this dimension is finite, the root manifold is closed. An isolated eigenvalue whose multiplicity is finite is called a *normal eigenvalue*. In the general case of a bounded linear operator  $A$  the linear manifold  $L_{\lambda_0}$  is not closed. If, however,  $L_{\lambda_0}$  happens to be closed, it is called a *root subspace*.

Let  $R_\lambda = (A - \lambda E)^{-1}$  be the resolvent of the operator  $A$  and  $\sigma(A)$  the spectrum of the operator  $A$ , i.e., the set of points  $\lambda$  where no bounded operator  $R_\lambda$  exists. We introduce the operator

$$P_{\sigma_1} = -\frac{1}{2\pi i} \int_{\Gamma_1} R_\lambda d\lambda,$$

where  $\Gamma_1$  is a closed rectifiable contour bounding some region  $G_1$  containing an isolated part  $\sigma_1$  of the spectrum of the operator  $A$ . The contour  $\Gamma_1$  consists of regular points of the operator  $A$  and has a positive orientation. The distance from  $\sigma_1$  to the closed set that is the complement of  $G_1$  is positive. We note the following properties of the operator  $G_{\sigma_1}$ :

a) If the spectrum  $\sigma(A)$  of the operator  $A$  is partitioned into two disjoint closed parts  $\sigma_1$  and  $\sigma_2$  ( $\sigma(A) = \sigma_1 \cup \sigma_2$ ,  $\sigma_1 \cap \sigma_2 = \emptyset$ ), then

$$P_{\sigma_1} + P_{\sigma_2} = E.$$

PROOF: Indeed let  $\Gamma_1$  and  $\Gamma_2$  be the boundaries of  $\sigma_1$  and  $\sigma_2$  respectively. Then

$$P_{\sigma_1} + P_{\sigma_2} = -\frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} R_\lambda d\lambda = -\frac{1}{2\pi i} \int_{|\lambda|=\|A\|+1} R_\lambda d\lambda,$$

since the spectrum of a bounded linear operator is contained in the disk  $C_z = \{z : |z| \leq \|A\|\}$ ; for the series  $R_\lambda = (A - \lambda E)^{-1} = -\frac{E}{\lambda} - \frac{A}{\lambda^2} - \dots - \frac{A^n}{\lambda^{n+1}} - \dots$  converges for  $|\lambda| > \|A\|$  uniformly on any compact subset of the complement to the disk  $C_z$ . Substituting the expansion just written for the resolvent into the integral and integrating termwise, we find from the relations

$$\int_{|\lambda|=a} \frac{d\lambda}{\lambda^n} = 0, \quad n \neq 1, \quad \int_{|\lambda|=a} \frac{d\lambda}{\lambda} = 2\pi i, \quad a > 0,$$

that

$$\begin{aligned} -\frac{1}{2\pi i} \int_{|\lambda|=\|A\|+1} R_\lambda d\lambda &= -\frac{1}{2\pi i} \int_{|\lambda|=\|A\|+1} \left[ -\sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}} \right] d\lambda \\ &= \sum_{n=0}^{\infty} \frac{A^n}{2\pi i} \int_{|\lambda|=\|A\|+1} \frac{d\lambda}{\lambda^{n+1}} = E, \end{aligned}$$

where  $A^0 = E$  and the order of integration and summation can be interchanged because of the uniform convergence (in the operator norm) of the series for the resolvent. ■

b) The operator  $P_{\sigma_1}$  is a projection (such operators are also called parallel projections), i.e.,  $P_{\sigma_1}^2 = P_{\sigma_1}$ ,  $P_{\sigma_1} = -\frac{1}{2\pi i} \int_{\Gamma_1} R_\lambda d\lambda$ , and  $\Gamma_1$  contains a part  $\sigma_1$  of the spectrum of the operator  $A$ . Moreover this part lies at a positive distance from the remainder of the spectrum.



PROOF: Indeed, consider a contour  $\Gamma'$  containing the set  $\sigma_1$  that lies entirely inside  $\Gamma_1$  and consists also of regular points of the operator  $A$ . Such a contour exists, by our assumptions on  $\sigma_1$ . Then by the Cauchy integral formula

$$P_{\sigma_1} = -\frac{1}{2\pi i} \int_{\Gamma'} R_\lambda d\lambda.$$

Consequently

$$P_{\sigma_1}^2 = P_{\sigma_1} P_{\sigma_1} = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} R_\lambda d\lambda \cdot \int_{\Gamma'} R_z dz = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma'} R_\lambda R_z d\lambda dz.$$

Using the Hilbert identity for the resolvent

$$R_\lambda - R_z = (\lambda - z) R_\lambda R_z,$$

one can write this last integral in the form

$$\begin{aligned} P_{\sigma_1}^2 &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma'} \frac{R_\lambda - R_z}{\lambda - z} d\lambda dz \\ &= \frac{1}{(2\pi i)^2} \left[ \int_{\Gamma_1} \int_{\Gamma'} \frac{R_\lambda}{\lambda - z} d\lambda dz - \int_{\Gamma_1} \int_{\Gamma'} \frac{R_z}{\lambda - z} d\lambda dz \right] \\ &= \frac{1}{(2\pi i)^2} \left[ \int_{\Gamma_1} R_\lambda d\lambda \int_{\Gamma'} \frac{dz}{\lambda - z} - \int_{\Gamma'} R_z dz \int_{\Gamma_1} \frac{d\lambda}{\lambda - z} \right] = -\frac{1}{2\pi i} \int_{\Gamma'} R_z dz = P_{\sigma_1}, \end{aligned}$$

i.e.,  $P_{\sigma_1}$  is a projection. ■

We remark that in this argument we have used the equality

$$\int_{\Gamma'} \frac{dz}{\lambda - z} = 0$$

(since the point  $\lambda$  lies outside the contour  $\Gamma'$ ) and the equality

$$\int_{\Gamma_1} \frac{d\lambda}{\lambda - z} = 2\pi i.$$

c) Let  $H_{\sigma_1} = P_{\sigma_1} H$ . Then  $H_{\sigma_1}$  is invariant under the operator  $A$ , i.e.,

$$P_{\sigma_1} A = A P_{\sigma_1}.$$

PROOF: This equality follows from the relation  $R_\lambda A = AR_\lambda$ . ■

d) If  $A_{\sigma_1}$  is the restriction of the operator  $A$  to  $H_{\sigma_1}$  and  $\sigma(A_{\sigma_1})$  is the spectrum of the operator  $A_{\sigma_1}$ , then  $\sigma(A_{\sigma_1}) \subset \sigma_1$ .

PROOF: Indeed, let  $\xi \notin \sigma_1$ . Then

$$(A - \xi E)R_\lambda = (A - \lambda E + (\lambda - \xi)E)R_\lambda = E + (\lambda - \xi)R_\lambda.$$

We rewrite this equality in the form

$$(A - \xi E) \cdot \frac{R_\lambda}{\lambda - \xi} = \frac{E}{\lambda - \xi} + R_\lambda.$$

We choose a contour  $\Gamma_1$  enclosing  $\sigma_1$  and such that the point  $\xi$  remains on the outside of the region enclosed by  $\Gamma_1$ . We integrate the last equality after multiplying by  $-\frac{1}{2\pi i}$ . We obtain

$$-\frac{1}{2\pi i}(A - \xi E) \int_{\Gamma_1} \frac{R_\lambda}{\lambda - \xi} d\lambda = \frac{E}{2\pi i} \int_{\Gamma_1} \frac{d\lambda}{\lambda - \xi} - \frac{1}{2\pi i} \int_{\Gamma_1} R_\lambda d\lambda = R_{\sigma_1},$$

since

$$\int_{\Gamma_1} \frac{d\lambda}{\lambda - \xi} = 0.$$

Thus

$$(A_{\sigma_1} - \xi E)B_{\sigma_1} = P_{\sigma_1},$$

where  $B_{\sigma_1} = -\frac{1}{2\pi i} \int_{\Gamma_1} \frac{R_\lambda}{\lambda - \xi} d\lambda$ , i.e., because all the operators commute, it

follows that the operator  $R_{\xi\sigma_1}$ —the resolvent of the operator  $A_{\sigma_1}$ —exists ( $\xi$  lies outside the region enclosed by the contour  $\Gamma_1$ ) and is the restriction of the operator  $-\frac{1}{2\pi i} \int_{\Gamma_1} \frac{R_\lambda}{\lambda - \xi} d\lambda$  to  $H_{\sigma_1}$ . It follows from what has been said

that  $\sigma(A_{\sigma_1})$  coincides with the part of the spectrum of the operator  $A$  that lies inside the region  $G$  enclosed by the contour  $\Gamma_1$ , i.e., coincides with  $\sigma_1$ . We remark that in our reasoning the set  $\sigma_1$  may also be empty. ■

Let  $A$  be a completely continuous operator in the Hilbert space  $H$ . Then outside an arbitrarily small circle with center at the origin the spectrum of

the operator  $A$  consists of a finite number of eigenvalues. Let  $\lambda_0$  be an arbitrary nonzero eigenvalue of the operator  $A$ . By what has been said above there exists a circle  $\Gamma_{\lambda_0}$  with center at  $\lambda_0$  such that  $\lambda_0$  is the unique point of the spectrum of the operator  $A$  inside this circle. We introduce the corresponding projection  $P_{\lambda_0} = -\frac{1}{2\pi i} \int_{\Gamma_{\lambda_0}} R_{\lambda} d\lambda$ .

e) The projection  $P_{\lambda_0}$ , where  $\lambda \neq 0$  is a point of the spectrum (i.e., an eigenvalue) of the completely continuous operator  $A$ , is finite-dimensional, i.e.,  $H_{\lambda_0} = P_{\lambda_0} H$  is a finite-dimensional subspace.

PROOF: In fact any point  $\lambda_0 \neq 0$  belonging to the spectrum  $\sigma(A)$  of the operator  $A$  forms an isolated part of the spectrum. We separate  $\lambda_0$  from the remaining points of the spectrum by a small circle  $\Gamma$  and write the equality

$$R_{\lambda} = -\frac{1}{\lambda} [(A - \lambda E) - A] R_{\lambda} = \frac{-E}{\lambda} + \frac{1}{\lambda} A R_{\lambda}.$$

Then

$$P_{\lambda_0} = -\frac{1}{2\pi i} \int_{\Gamma} R_{\lambda} d\lambda = \frac{1}{2\pi i} \int_{\Gamma} \frac{d\lambda}{\lambda} - \frac{A}{2\pi i} \int_{\Gamma} \frac{R_{\lambda}}{\lambda} d\lambda = A \cdot B,$$

where  $B = -\frac{1}{2\pi i} \int_{\Gamma} \frac{R_{\lambda}}{\lambda} d\lambda$ . Since  $A$  is a completely continuous operator and  $B$  is bounded, we find that  $P_{\lambda_0}$  is completely continuous. We shall show that the subspace  $H_{\lambda_0} = P_{\lambda_0} H$  is finite-dimensional. Supposing the contrary, we would construct an infinite orthonormal system  $\{\varphi_k\}$  in it using the Gram-Schmidt procedure. Then on the one hand we would have  $P_{\lambda_0} \varphi_k = \varphi_k$  by the invariance of  $H_{\lambda_0}$  under  $P_{\lambda_0}$ . That is, because  $P_{\lambda_0}$  is completely continuous, we could choose a convergent subsequence of the sequence  $\varphi_k$ ; yet on the other hand  $\|\varphi_k - \varphi_i\|^2 = 2$ , so that this is impossible. The contradiction so obtained proves that  $H_{\lambda_0}$  is finite-dimensional. ■

The following lemma holds.

LEMMA 3. Let  $A$  be a completely continuous operator and  $\lambda_0$  a nonzero eigenvalue of  $A$ . Then the root subspace  $L_{\lambda_0}$  is finite-dimensional and coincides with the space  $H_{\lambda_0} = P_{\lambda_0} H$ .

PROOF: We shall prove that  $L_{\lambda_0} = H_{\lambda_0}$ . We first of all remark that  $L_{\lambda_0} \supset H_{\lambda_0}$ . In fact, since the restriction  $A_{\lambda_0}$  of the operator  $A$  to  $H_{\lambda_0}$  has in its spectrum only the point  $\lambda_0$  (property c)) and the space  $H_{\lambda_0}$  is



finite-dimensional (property d)) it follows that  $H_{\lambda_0}$  can be decomposed into a finite number of subspaces on each of which  $A$  is a Jordan block of the form

$$\begin{pmatrix} \lambda_0 & 1 & 0 & \dots & 0 \\ 0 & \lambda_0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_0 \end{pmatrix},$$

i.e.,  $H_{\lambda_0}$  is annihilated by some power of the operator  $A - \lambda_0 E$ , and consequently  $H_{\lambda_0} \subset L_{\lambda_0}$ . Now suppose  $L_{\lambda_0} \neq H_{\lambda_0}$ . Then there exists a vector  $h \in L_{\lambda_0}$  with  $h \notin H_{\lambda_0}$ . We write the decomposition of the vector  $h$ :  $h = h_1 + h_2$ ,  $h_1 \in H_{\lambda_0}$  and  $h_2 \in L_{\lambda_0} \ominus H_{\lambda_0} = H_{\lambda_0}^\perp$ —the orthogonal complement of  $H_{\lambda_0}$  in  $L_{\lambda_0}$ . There exists a natural number  $s_1$  such that  $(A - \lambda_0 E)^{s_1} h = 0$ , since  $h$  is a root vector. From what was proved above there exists a natural number  $s_2$  such that  $(A - \lambda_0 E)^{s_2} h_1 = 0$ , since  $h_1 \in H_{\lambda_0}$ . If we take  $s = \max(s_1, s_2)$ , then  $(A - \lambda_0 E)^s h_2 = 0$ . We choose the smallest number  $s$  such that  $(A - \lambda_0 E)^s h_2 = 0$ . We find that  $(A - \lambda_0 E)^{s-1} h_2 \neq 0$ . But then  $f = (A - \lambda_0 E)^{s-1} h_2$  is an eigenvector of the operator  $A$ , since

$$(A - \lambda_0 E)f = (A - \lambda_0 E)^s h_2 = 0.$$

Now the spectrum of the operator  $A$  in  $H_{\lambda_0}^\perp$  consists of the set  $\sigma(A) \setminus \{\lambda_0\}$ . Thus, having arranged that  $h_2 \neq 0$ ,  $h_2 \in H_{\lambda_0}^\perp$ , we have constructed an eigenvector  $f \neq 0$  out of  $h_2$  corresponding to the given eigenvalue  $\lambda_0$  and thus we have arrived at a contradiction. Therefore  $L_{\lambda_0} = H_{\lambda_0}$ , and the space  $L_{\lambda_0}$  is finite-dimensional since  $H_{\lambda_0}$  is. In other words, the nonzero part of the spectrum of a completely continuous operator consists of normal eigenvalues. ■

**DEFINITION 12.** A bounded linear operator  $A$  on a Hilbert space  $H$  is *finite-dimensional* if its image  $AH$  is a finite-dimensional subspace of  $H$ . The dimension of this subspace is called the *dimension* of the operator  $A$  and is denoted  $\dim A$ .

**REMARK.** Taking account of this definition, one can rephrase part of Lemma 3 by saying that the operator  $P_{\lambda_0}$  is finite-dimensional and its dimension is equal to the algebraic multiplicity of the eigenvalue  $\lambda_0$ .

The following properties of finite-dimensional operators can be proved easily by proceeding directly from the definition:

1) Let  $A$  be a bounded operator on a Hilbert space  $H$  and let  $B$  be a finite-dimensional operator. Then the operators  $AB$  and  $BA$  are finite-dimensional and their dimensions do not exceed the dimension of  $B$ .

2) Let  $A$  and  $B$  be finite-dimensional operators on the Hilbert space  $H$ . Then the operator  $A + B$  is also finite-dimensional and  $\dim(A + B) \leq \dim A + \dim B$ .

We now resume the study of the properties of the projection  $P_{\sigma_1}$ . We shall show how it is possible to use the resolvent to construct a functional calculus of an operator  $A$  that is bounded on the invariant subspace  $P_{\sigma_1}H$ .

LEMMA 4. The following equalities hold:

$$A^n P_{\sigma_1} = -\frac{1}{2\pi i} \int_{\Gamma_1} \lambda^n R_\lambda d\lambda.$$

PROOF: We use the identity

$$A^n - \lambda^n E = (A - \lambda E)(A^{n-1} + \lambda A^{n-2} + \cdots + \lambda^{n-1} E).$$

Multiplying both sides of this equality by  $-\frac{1}{2\pi i} R_\lambda$  and integrating over a contour  $\Gamma_1$  enclosing a part  $\sigma_1$  of the spectrum, we obtain

$$-\frac{1}{2\pi i} \int_{\Gamma_1} [R_\lambda (A^n - \lambda^n E)] d\lambda = -\frac{1}{2\pi i} \int_{\Gamma_1} (A^{n-1} + \lambda A^{n-2} + \cdots + \lambda^{n-2} E) d\lambda.$$

We remark that the integrand on the right-hand side of this equality is an analytic operator-valued function inside the contour  $\Gamma_1$ . Moreover, since the integration is around a closed contour, we conclude that the expression on the left-hand side is equal to zero, from which, by the fact that the operators  $A$  and  $R_\lambda$  commute, the assertion to be proved follows. ■

Using Lemma 4 and the additivity of the integral, we easily prove the relation

$$p(A)P_{\sigma_1} = -\frac{1}{2\pi i} \int_{\Gamma_1} p(\lambda) R_\lambda d\lambda,$$

where  $p(\lambda)$  is an arbitrary polynomial in  $\lambda$ . Furthermore, using the same reasoning as in the construction of the functional calculus in the proof of the spectral theorems, one can enlarge the class of functions  $f(\lambda)$  satisfying the equality

$$f(A)P_{\sigma_1} = -\frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda) R_\lambda d\lambda.$$



We use the results we have obtained to study the analytic properties of the resolvent of a completely continuous operator in the neighborhood of an eigenvalue.

Let  $A$  be a completely continuous operator on the Hilbert space  $H$  and  $\lambda_0$  a nonzero eigenvalue of  $A$ . Then by Lemma 4 we have

$$(A - \lambda_0 E)^n P_{\lambda_0} = -\frac{1}{2\pi i} \int_{\Gamma_{\lambda_0}} (\lambda - \lambda_0)^n R_{\lambda} d\lambda.$$

As already noted, the operator-valued function  $R_{\lambda}$  is an analytic function in a neighborhood of  $\lambda_0$  and consequently admits a Laurent expansion in a neighborhood of this point:

$$R_{\lambda} = A(\lambda) + \frac{B_1}{\lambda - \lambda_0} + \frac{B_2}{(\lambda - \lambda_0)^2} + \cdots + \frac{B_m}{(\lambda - \lambda_0)^m} + \cdots,$$

where  $A(\lambda)$  is the regular part of the Laurent series and  $B_k$  are constant operators. Applying Lemma 4 we find that

$$B_m = \frac{1}{2\pi i} \int_{\Gamma_{\lambda_0}} (\lambda - \lambda_0)^{m-1} R_{\lambda} d\lambda = -(A - \lambda_0 E)^{m-1} P_{\lambda_0}.$$

In particular the residue of the resolvent of a completely continuous operator in the neighborhood of a non-zero eigenvalue  $\lambda_0$  is  $-P_{\lambda_0}$ . Moreover, since the operator  $P_{\lambda_0}$  is finite-dimensional, it follows from properties 1) and 2) of finite-dimensional operators shown above that the operators  $B_m$  are also finite-dimensional, and their dimensions do not exceed the dimension of the operator  $P_{\lambda_0}$ , which equals the dimension of the corresponding root subspace  $L_{\lambda_0}$ . We shall show that from some index  $m_0$  on all the operators  $B_m$  are identically zero, i.e., the point  $\lambda_0$  is a pole of the operator-valued function  $R_{\lambda}$ .

In fact the operator  $P_{\lambda_0}$  projects the space  $H$  onto the root subspace  $L_{\lambda_0}$  which is invariant with respect to the operator  $A$ . Then the restriction of the operator  $A$  to this subspace is a linear operator on a finite-dimensional space, and its spectrum consists of the single point  $\lambda_0$ . Then on  $L_{\lambda_0}$  the operator  $A$  can be reduced to Jordan form, which in the present case has



the form

$$\begin{pmatrix} \lambda_0 & 1 & & & & \\ & \ddots & 1 & & & \\ & & \lambda_0 & & & \\ & & & \ddots & 1 & \\ & & & & \lambda_0 & \\ & & & & & \ddots \\ & & & & & & \ddots \end{pmatrix}$$

It is from this representation that the assertion regarding the operators  $B_m$  follows, and one can take as  $m_0$  the length of the longest Jordan chain corresponding to the eigenvalue  $\lambda_0$ .

We introduce the notation

$$L = \bigcup_{\lambda_k \neq 0} L_{\lambda_k}, \quad Z = H \ominus \bar{L} = L^\perp,$$

where  $L_{\lambda_k}$  are the root manifolds corresponding to the eigenvalues  $\lambda_k$ ,  $Z$  is the orthogonal complement to  $\bar{L}$  in  $H$ , and  $L$  is the linear span of the root manifolds  $L_{\lambda_k}$ .

**DEFINITION 13.** The operator  $A$  is called a *Volterra operator* if it is completely continuous and has no non-zero eigenvalues.

**LEMMA 5.** Let  $Q$  be the orthogonal projection on the subspace  $Z$ , i.e.,  $QH = Z$ . Let  $Q^2 = Q$ ,  $Q = Q^*$ , and let  $A$  be a completely continuous operator. Then the operator  $QA^*Q$  is a Volterra operator.

**PROOF:** Suppose  $QA^*Q$  is not a Volterra operator. Since the operator  $QA^*Q$  is completely continuous, there exists a vector  $y_0 \neq 0$ ,  $y_0 \in Z$  such that  $A^*y_0 = \lambda_0 y_0$ , and  $\lambda_0 \neq 0$ .\* Then

$$0 = (h, (A^* - \lambda_0)y_0) = (h, 0) = ((A - \bar{\lambda}_0)h, y_0)$$

for any vector  $h \in H$ . In other words the vector  $y_0 \perp (A - \bar{\lambda}_0)H$ . On the other hand  $y_0 \in Z$ , i.e.,  $y_0 \perp L_{\bar{\lambda}_0}$ . We denote by  $\{\lambda_0\}'$  the complement of the point  $\bar{\lambda}_0$  relative to the spectrum of the operator  $A$ . Then the operator

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\*The subspace  $\bar{L}$  is invariant with respect to the operator  $A$ . Consequently the subspace  $Z$  is invariant with respect to  $A^*$  and the restriction of  $A^*$  to  $Z$  coincides with  $QA^*Q$ .

$A - \bar{\lambda}_0 E$  is invertible on  $L_{\{\lambda_0\}'}$ . Thus  $y_0 \perp L_{\{\lambda_0\}'}$ . It was shown above (cf. property a)) that  $H = L_{\bar{\lambda}_0} + L_{\{\lambda_0\}'}$ . It then follows from the relations  $y_0 \perp L_{\bar{\lambda}_0}$  and  $y_0 \perp L_{\{\lambda_0\}'}$  that  $y_0 = 0$ , contrary to hypothesis. Hence the operator  $QA^*Q$  is a Volterra operator. ■

It was shown in Sec. 4.2 that a self-adjoint (symmetric) completely continuous operator can be reduced to diagonal form. We shall now prove the following lemma on the reduction of an arbitrary completely continuous operator to triangular form.

**LEMMA 6 (Schur).** *Let  $A$  be a completely continuous operator and  $L = \bigcup_{\lambda_k \neq 0} L_{\lambda_k}$ . Then there exists a complete orthonormal system  $\{\varphi_i\}$  in  $L$  such that when it is chosen as a basis, the matrix of the operator  $A$  assumes triangular form, i.e.*

$$A\varphi_i = \alpha_{i1}\varphi_1 + \alpha_{i2}\varphi_2 + \cdots + \alpha_{ii}\varphi_i,$$

and

$$(A\varphi_i, \varphi_j) = \alpha_{ij}, \quad \alpha_{ii} = (A\varphi_i, \varphi_i) = \lambda_i(A).$$

Hence the diagonal contains the eigenvalues of the matrix.

**PROOF:** In each of the subspaces  $L_{\lambda_k}$  we choose a Jordan basis  $\{\varphi_i^{(k)}\}$ :

$$A\varphi_i^{(k)} = \lambda_i \varphi_i^{(k)} + \varphi_{i-1}^{(k)}$$

or

$$A\psi_i^{(k)} = \lambda_i \psi_i^{(k)},$$

if  $\psi_i^{(k)}$  is an eigenvector of the operator  $A$ . Enumerating the elements of all these Jordan bases, we obtain a countable number of vectors any finite subset of which is linearly independent. Orthogonalizing and normalizing this countable system, we obtain the required orthonormal system.

### 3.4. Differential Operators

Differentiation operators, as we have already noted, are not bounded. To study them one usually resorts to their inverse operators, which are bounded and in many cases even completely continuous. However a direct study of differentiation operators (without recourse to the inverse) also has many advantages. We shall touch on only a few of the most general concepts connected with differentiation operators.

A linear differential operator is an expression of the form

$$l(y) = p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y,$$

where the functions  $p_0(x), p_1(x), \dots, p_n(x)$  are the coefficients, and the number  $n$  is the order of the differential operator. The functions  $[p_0(x)]^{-1}, p_1(x), \dots, p_n(x)$  are assumed continuous; in some cases additional hypotheses are imposed on them. If we denote by  $C^n[a, b]$  the set of functions  $y(x)$  having continuous derivatives up to order  $n$  inclusive, then for any function  $y \in C^n[a, b]$  the differential operator  $l(y)$  is defined on the closed interval  $[a, b]$ .

Let

$$y(a), y'(a), \dots, y^{(n-1)}(a), y(b), y'(b), \dots, y^{(n-1)}(b)$$

be the values of the function  $y(x)$  and its derivatives at the points  $a$  and  $b$  respectively.

We denote by  $U(y)$  a linear form in these variables having the form

$$U(y) = \alpha_0 y(a) + \alpha_1 y'(a) + \dots + \alpha_{n-1} y^{(n-1)}(a) \\ + \beta_0 y(b) + \beta_1 y'(b) + \dots + \beta_{n-1} y^{(n-1)}(b).$$

If several such forms are defined, then the equalities  $U_r(y) = 0$  are called *boundary conditions*. In general it makes sense to consider only linearly independent forms  $U_r(y)$ .

The homogeneous boundary-value problem for a given differential expression  $l(y)$  is defined as the problem of determining a function  $y$  in  $C^n[a, b]$  satisfying the equalities

$$l(y) = 0, \quad U_r(y) = 0, \quad r = 1, 2, \dots, m.$$

In what follows we shall consider only the case  $m = n$ , where  $n$  is the order of  $l(y)$ .

We shall investigate the conditions under which a homogeneous boundary-value problem has nontrivial solutions  $y(x)$ . Let  $y_1, y_2, \dots, y_n$  be linearly independent solutions of the differential equation  $l(y) = 0$ . As is known, the general solution can be written in the form

$$y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n.$$

We determine the constants  $C_1, C_2, \dots, C_n$  so that  $y$  is a solution of the boundary-value problem, i.e., also satisfies the boundary conditions  $U_r(y) = 0$  for  $r = 1, 2, \dots, n$ . We easily deduce that the homogeneous boundary-value problem has nontrivial solutions if and only if the determinant of the matrix

$$U = \begin{pmatrix} U_1(y_1) & \dots & U_1(y_n) \\ \dots & \dots & \dots \\ U_n(y_1) & \dots & U_n(y_n) \end{pmatrix}$$



is zero.

It is particularly interesting to investigate the case of the boundary-value problem when the coefficients of the differential operator and the boundary forms are analytic functions of some numerical parameter  $\lambda$  and are regular in some region  $D$ .

The values of the parameter  $\lambda$  for which the homogeneous boundary-value problem has a nontrivial solution are called the *eigenvalues* of the problem and the solutions corresponding to them are called the *eigenfunctions* or *eigenvectors*. In this case the determinant of the matrix (which we denote by  $\Delta(\lambda)$ ) is an analytic function of the parameter  $\lambda$  in the region  $D$ . Indeed, the linearly independent solutions  $y_1(x, \lambda), y_2(x, \lambda), \dots, y_n(x, \lambda)$  can always be chosen as solutions of the Cauchy problem satisfying the conditions

$$y_j^{(r-1)}(a, \lambda) = \begin{cases} 0 & \text{for } j \neq r, \\ 1 & \text{for } j = r, \end{cases}$$

at the point  $a$  for  $r, j = 1, 2, \dots, n$ . In this case the solutions  $y_j(x, \lambda)$ ,  $j = 1, 2, \dots, n$ , will be analytic in functions in the region by the theorem on analytic dependence of the solutions on a parameter, and then the function  $\Delta(\lambda)$ , the characteristic determinant of the problem, will also be analytic in the same region.

As already stated, the eigenvalues of a homogeneous boundary-value problem must be zeros of the determinant of the matrix  $U$ , i.e., zeros of the function  $\Delta(\lambda)$ . Since  $\Delta(\lambda)$  is an analytic function of the parameter  $\lambda$  in the region  $D$ , two possibilities may arise:

1.  $\Delta(\lambda) \equiv 0$  in the region  $D$ ; then every number  $\lambda \in D$  is an eigenvalue of the problem.

2.  $\Delta(\lambda) \not\equiv 0$ ; then the region  $D$  contains at most a countable number of eigenvalues and the set of eigenvalues has no limit points in the interior of  $D$ . In particular if the coefficients of the differential operator  $l(y)$  and the boundary forms  $U_r(y)$  are entire functions of  $\lambda$ , then  $\Delta(\lambda)$  is also an entire function; consequently in this case by the uniqueness theorem for analytic functions the eigenvalues cannot have a finite limit point (unless  $\Delta(\lambda) \equiv 0$ ).

Thus the problem of finding the eigenvalues of a boundary-value problem has been reduced to the problem of finding the zeros of the function  $\Delta(\lambda)$ , i.e., to solving the equation  $\Delta(\lambda) = 0$ . In the general case it is not possible to obtain precise information on the location of these zeros. However, one most frequently studies particular cases of dependence of the coefficients of  $l(y)$  and  $U_r(y)$  on the parameter  $\lambda$ . The simplest and best studied problem is the following:  $l(y) = \lambda y$ ,  $U_r(y) = 0$ , where the coefficients of the expression  $l(y)$  and the forms  $U_r(y)$  are independent of  $\lambda$ . Polynomial

dependence of the coefficients of the problem on the parameter  $\lambda$  is also frequently studied. If, however, we consider a complicated dependence of the coefficients and boundary forms on the parameter  $\lambda$  and we wish to obtain precise information about the eigenvalues  $\lambda_n$  or the eigenfunctions  $y_n(x)$ , we must impose additional conditions on the coefficients of the boundary-value problem. These conditions are most often such that, for example, the function  $\Delta(\lambda)$  admits a representation

$$\Delta(\lambda) = \sum_{k=0}^{N-1} e^{\alpha_k \lambda} P_{k,h}(\lambda)$$

for each integer  $h \geq 0$ , where  $\alpha_k$  are complex constants and

$$P_{k,h}(\lambda) = \lambda^{n_k} \sum_{\nu=0}^h \beta_{\nu}^{(k)} \lambda^{-\nu} + O(\lambda^{n_k-h})$$

as  $\lambda \rightarrow \infty$ . Here  $n_k$  are integers and for sufficiently large values of  $|\lambda|$  the functions  $P_{k,h}(\lambda)$  are analytic in some sectors containing the origin and covering the entire  $\lambda$ -plane.

Now it is possible to find asymptotic estimates for the zeros of a function  $\Delta(\lambda)$  of this form. Asymptotic formulas for the zeros of such functions  $\Delta(\lambda)$  can be found, for example, by the method of successive approximations. The corresponding formulas will be presented in the next chapter.

### EXAMPLES

1. Find the eigenvalues and eigenfunctions of the boundary-value problem

$$\begin{cases} -y'' = \lambda y, \\ U_1(y) = y(0) = 0, \\ U_2(y) = y(\pi) = 0. \end{cases}$$

Let  $\lambda = s^2$ . Then the general solution of this equation can be written in the form  $y = C_1 \sin sx + C_2 \cos sx$ . Furthermore

$$\Delta(\lambda) = \begin{vmatrix} 0 & 1 \\ \sin s\pi & \cos s\pi \end{vmatrix} = -\sin s\pi.$$

Therefore  $\lambda_n = s_n^2 = n^2$ ,  $y_n(x) = C \sin nx$ ,  $n = 1, 2, \dots$ , where  $C$  is a constant chosen usually so as to normalize the eigenfunctions.

2. Consider the problem

$$\begin{aligned} -y'' &= \lambda y, & U_1(y) &= y(0) - y(1) = 0, \\ U_2(y) &= y'(0) + y'(1) = 0. \end{aligned}$$

Let  $\lambda = s^2$  and  $y_1 = \sin sx$ ,  $y_2 = \cos sx$ . Then

$$\Delta(\lambda) = \begin{vmatrix} -\sin s & 1 - \cos s \\ s(1 + \cos s) & -s \sin s \end{vmatrix} = s(\sin^2 s + \cos^2 s - 1) \equiv 0,$$

i.e., every  $\lambda$  is an eigenvalue. The eigenfunctions in this case are written in the form  $C \cos s\left(x + \frac{1}{2}\right)$ .

We remark that in these examples we have chosen the fundamental system of solutions  $y_1 = \sin sx$ ,  $y_2 = \cos sx$ . We could have chosen a fundamental system of functions that are entire in  $s$  (for each fixed  $x$ ):

$$\begin{aligned} z_1 &= \frac{1}{s} \sin sx, & z_1(0) &= 0, & z_1'(0) &= 1; \\ z_2 &= \cos sx, & z_2(0) &= 1, & z_2'(0) &= 0. \end{aligned}$$

3. Consider the boundary-value problem:

$$-y^{iv} = \lambda y'', \quad y''(0) = y'''(0) = y(1) = y'(1) = 0.$$

For  $\lambda = 0$  the equation  $y^{iv} = 0$  has the general solution  $y = C_0 + C_1x + C_2x^2 + C_3x^3$ ; under the given boundary conditions we find that  $y \equiv 0$  and so  $\lambda = 0$  is not an eigenvalue. If  $\lambda \neq 0$ , the general solution of the equation is  $y = C_1 + C_2x + C_3 \sin xs + C_4 \cos sx$ . From the condition  $y''(0)$  we have  $C_4 = 0$  and the conditions  $y'''(0) = 0$ ,  $y'(1) = 0$ ,  $y(1) = 0$  require that  $C_3$ ,  $C_2$ , and  $C_1$  respectively be equal to zero, i.e., this problem has neither eigenvalues nor eigenfunctions.

4. Let

$$\begin{aligned} l(y) &\equiv y'' + 2a\lambda y' + b\lambda^2 y, \\ U_1(y) &= y(0) = 0, & U_2(y) &= y(\pi) = 0, \end{aligned}$$

where  $a$  and  $b$  are certain numbers with  $a^2 - b = c < 0$ . We easily find that in this case the eigenvalues are  $\lambda_n = \gamma^{-1}n$ ,  $\gamma = \sqrt{|c|}$ , and the eigenfunctions are  $y_n = e^{-anz} \sin \gamma nx$ ,  $n = \pm 1, \pm 2, \dots$ .



5. As an example let us consider in more detail the differential expression

$$l(y) \equiv -y'' + p(x)y$$

on the closed interval  $[0, \pi]$ , where we assume the function  $p(x)$  is continuous on  $[0, \pi]$ . The differential expression  $l(y)$  generates different differential operators on different spaces.

If we consider the space  $C[0, \pi]$ , the operator  $L$  corresponding to the differential expression  $l(y)$  can be defined on the space  $D_L$  of twice continuously differentiable functions by setting  $Ly = l(y)$  for  $y \in D_L$ . The space  $D_L$  can be restricted by considering various boundary conditions:  $y(0) = y(\pi) = 0$  (zero boundary conditions) or  $y'(0) = y'(\pi)$  (elastic boundary conditions).

On the domain  $D_L$  the operator  $L$  is maximal in  $C[0, \pi]$  and the other operators are now restrictions of it.

The operators  $L$  on  $D_L$  are noninvertible, since  $Ly = 0$  has two linearly independent solutions belonging to  $C[0, \pi]$ . The operator  $L_1$  given by

$$l(y) \equiv -y'' + p(x)y, \quad y(0) = y(\pi) = 0$$

is possibly invertible under certain additional conditions. The operator  $L_2$

$$l(y) \equiv -y'' + p(x)y, \quad y(0) = y'(\pi) = 0$$

is always invertible, since the Cauchy problem has a unique solution. The inverse to the operator  $L_1$

$$Ly \equiv -y'' + p(x)y, \quad y(0) = y(\pi) = 0$$

is an integral operator

$$L^{-1}f = \int_0^\pi G(x, \xi)f(\xi) d\xi,$$

where  $G(x, \xi)$  is the Green's function for zero boundary conditions

$$G(x, \xi) = \frac{1}{W} \begin{cases} y_1(x)y_2(\xi), & x \leq \xi, \\ y_2(x)y_1(\xi), & x \geq \xi, \end{cases}$$

where  $y_1$  and  $y_2$  are nonzero solutions of the equation  $Ly = 0$  such that  $y_1(0) = 0$ ,  $y_2(\pi) = 0$ , and  $W$  is their Wronskian. (If  $y_1(\pi) \neq 0$ , then  $W \neq 0$  and the Green's function is defined.)

6. Let us consider again (cf. Sec. 4.2.13) the "simplest" operator

$$l(y) \equiv -y'', \quad y(0) = y(\pi) = 0$$

and study its resolvent. It is obvious that the eigenvalues of the operator are  $\lambda_n = n^2$  and the eigenfunctions are  $\sin nx$ ,  $n = 1, 2, 3, \dots$ .

The resolvent  $R_\lambda$  is an integral operator whose kernel coincides with the Green's function of the equation  $-y'' = \lambda y = s^2 y$  for zero boundary conditions.

It is not difficult to compute that

$$G(x, \xi, \lambda) = \frac{\sin sx \cdot \sin s(\pi - \xi)}{s \cdot \sin \pi s}, \quad x \leq \xi, \quad s^2 = \lambda,$$

and for  $x \geq \xi$  it is necessary to interchange  $x$  and  $\xi$  on the right-hand side of the expression for  $G$ .

In particular from the formula for the Green's function we find that the poles of the Green's function are the points  $s_n = n$ , i.e.,  $\lambda_n = n^2$  are the eigenvalues of this operator.

A detailed study of the behavior of the Green's function is usually carried out in the study of the completeness and basis properties of the eigenfunctions of the operator.

