Chapter 5

The Trace of an Operator

In this chapter we study questions connected with the concept of the trace of a linear operator. It is well-known that the sum of the diagonal elements of the matrix of a linear transformation in n-dimensional space, i.e., the trace of the matrix, is the sum of the eigenvalues of the operator, counted according to their multiplicities.

The question naturally arises: For which classes of operators does this fact hold good in a Hilbert space? The trace of an operator plays an important role in many parts of analysis in questions of the approximate computation of eigenvalues and in solving the inverse problem of spectral analysis. The study is also of independent interest.

1. THE TRACE THEOREM FOR AN OPERATOR IN n-DIMENSIONAL SPACE

We agree to call the sum of the eigenvalues of a linear operator in n-dimensional space the *spectral trace* and the sum of the diagonal elements of the matrix of the transformation the *matrix trace*.

THEOREM 1. *The matrix trace of a linear operator A in n-dimensional space is equal to its spectral trace.*

PROOF: Let the transformation A be given in some basis by the matrix $\|a_{ik}\|_{i,k=1}^{n}$. We shall show that the characteristic polynomial $\Delta(\lambda)$ of the transformation A—the determinant of the matrix $A - \lambda E$, where $A = \|a_{ik}\|_{i,k=1}^{n}$—is independent of the choice of basis. Indeed it is known that in a new basis the matrix A assumes the form $C^{-1}AC$, where C is the transition matrix to the new basis. Then

$$\begin{aligned}&\det\|C^{-1}AC-\lambda E\|=\det\|C^{-1}AC-\lambda C^{-1}EC\|\\&=\det\|C^{-1}(A-\lambda E)C\|=\det\|C^{-1}\|\cdot\det\|A-\lambda E\|\cdot\det\|C\|=\Delta(\lambda).\end{aligned}$$

We now compute $\Delta(\lambda)=\det\|A-\lambda E\|$ in terms of the elements of the matrix A. To calculate the coefficient of $(-\lambda)^k$ we must take a sum of determinants each of which is obtained by replacing k columns of the matrix $\|a_{ij}\|_{i,j=1}^n$ by k columns of the identity matrix. But each such determinant is a principal minor of order k of the matrix $\|a_{ij}\|_{i,j=1}^n$. Thus the characteristic polynomial $\Delta(\lambda)$ of the matrix A has the form

$$\Delta(\lambda)=(-1)^n(\lambda^n-S_1\lambda^{n-1}+S_2\lambda^{n-2}-\cdots\pm S_n),$$

where S_1 is the sum of the diagonal elements, S_2 is the sum of the principal minors of order 2, etc. The last term S_n is the determinant of the matrix of A. On the other hand S_1 is the sum of all the zeros of $\Delta(\lambda)$ (the eigenvalues of A) counted according to their multiplicities, as one can easily verify by calculating $\Delta(\lambda)$ in a basis that puts A in Jordan form. Thus the spectral trace of the operator $\sum_{i=1}^n\lambda_i$ is equal to its matrix trace $\operatorname{Sp}A=\sum_{i=1}^n a_{ii}$, i.e., we have the equality

$$\sum_{i=1}^n\lambda_i=\sum_{i=1}^n a_{ii},$$

which was to be proved. ■

2. NUCLEAR OPERATORS. THE TRACE THEOREM

2.1. The Trace Theorem for a Positive Nuclear Operator

We now resume our study of completely continuous operators. It is well known that for an operator A on $\mathbf{R}^n$ one can obtain the following representation:*

$$Af=(f,g_1)f_1+(f,g_2)f_2+\cdots+(f,g_n)f_n,$$

where $\{f_n\}$ is some basis of $\mathbf{R}^n$ and $\{g_k\}_{k=1}^n$ is some finite system of vectors independent of f.

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*In fact if $\{\varphi_j\}$ is the system biorthogonal to f_j, the equality $Af=\sum_{j=1}^n c_jf_j$ implies that $c_j=(Af,\varphi_j)=(f,A^*\varphi_j)=(f,g_j)$.

THEOREM 2. *Let A be an arbitrary completely continuous operator in the Hilbert space H, and let H_0 be its nullspace. It is possible to exhibit two orthonormal systems of vectors $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ and a monotonically decreasing sequence of positive numbers $\{s_k\}_{k=1}^{\infty}$, $s_k \to 0$ as $k \to \infty$, such that for any vector $f \in H$ the following expansions converge in the norm of H:*

$$f = f_0 + (f, f_1)f_1 + (f, f_2)f_2 + \cdots, \quad f_0 \in H_0,$$
$$Af = s_1(f, f_1)g_1 + s_2(f, f_2)g_2 + \dots .$$

PROOF: If the operator A is symmetric and positive, then our assertions are a consequence of the reasoning carried out for completely continuous operators in the preceding chapter (Sec. 4.2.10). Indeed it was shown that for any vector $f \in H$ we have the equality $Af = \sum_{i=1}^{\infty} \lambda_i(f, f_i)f_i$. We denote the vector $f - \sum_{i=1}^{\infty}(f, f_i)f_i$ by f_0. Then $f = f_0 + \sum_{i=1}^{\infty}(f, f_i)f_i$ and $Af_0 = Af - \sum_{i=1}^{\infty}(f, f_i)Af_i = 0$. Consequently in this case the theorem is true, and one can set $s_i = \lambda_i$, $g_i = f_i$.

Let A be an arbitrary completely continuous operator. We form the operator $B = A^* A$. By the corollary to Theorem 1 of Sec. 4.2.2 the operator B is completely continuous and positive. We denote by λ_k the sequence of its nonzero eigenvalues ($\lambda_k \downarrow 0$), and by $\{f_k\}$ an orthonormal sequence of eigenvectors corresponding to the eigenvalues λ_k. Then we find that $\lambda_k = (Bf_k, f_k) = (A^* Af_k, f_k) = (Af_k, Af_k) = s_k^2 > 0$ and $s_k > 0, s_1 \geqslant s_2 \geqslant \cdots$.

The numbers s_i play an important role in the study of completely continuous operators; they are called the s-numbers of the operator A.

We remark that the nullspaces of the operators B and A coincide since for any $f_1, f_2 \in H$

$$(Af_1, Af_2) = (A^* Af_1, f_2) = (Bf_1, f_2).$$

Therefore, since any vector f has an expansion $f = f_0 + \sum_{k=1}^{\infty}(f, f_k)f_k$, it follows that

$$Af = \sum_{k=1}^{\infty}(f, f_k)Af_k = \sum_{k=1}^{\infty} s_k(f, f_k)g_k,$$

where we use the notation $Af_k = s_k g_k$.

Furthermore,

$$(Af_k, Af_i) = s_k \cdot s_i (g_k, g_i) = (Bf_k, f_i) = \lambda_k (f_k, f_i) = \lambda_k \delta_{ki}.$$

Therefore $(g_k, g_i) = \delta_{ki}$, i.e., the sequence $\{g_k\}_{k=1}^{\infty}$ is orthonormal. ■

Now let the operator A have a finite absolute norm $N(A)$. Then supplementing the system $\{f_k\}$ if necessary by a basis of the nullspace of the operator A to form an orthonormal basis $\{f'_k\}$ of the whole space, we have

$$N^2(A) = \sum_{k=1}^{\infty} \|Af'_k\|^2 = \sum_{k=1}^{\infty} (Af'_k, Af'_k)$$
$$= \sum_{k=1}^{\infty} (A^* Af'_k, f'_k) = \sum_{k=1}^{\infty} \lambda_k (f_k, f_k) = \sum_{k=1}^{\infty} s_k^2,$$

where f_k is an orthonormal system of eigenvectors of the operator $A^* A = B$, corresponding to nonzero eigenvalues.

DEFINITION 1. A completely continuous operator A is *of Schmidt class* if the series of squares of its s-numbers converges, i.e., if $\sum\limits_{k=1}^{\infty} s_k^2 < \infty$.

Thus* a completely continuous operator A is of Schmidt class if and only if it has a finite absolute norm.

DEFINITION 2. A completely continuous operator A is *nuclear* if the series of its s-numbers converges, i.e., if $\sum\limits_{k=1}^{\infty} s_k < \infty$.

THEOREM 3. *If A is a nuclear operator, then for any choice of an orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ of H the series $\sum\limits_{k=1}^{\infty} (A\varphi_k, \varphi_k)$ converges absolutely, the inequality $\sum\limits_{k=1}^{\infty} |(A\varphi_k, \varphi_k)| \leqslant \sum\limits_{k=1}^{\infty} s_k$ holds, and the sum of the series $\sum\limits_{k=1}^{\infty} (A\varphi_k \varphi_k)$, which we shall call the matrix trace of the operator A and denote* Sp A, *is independent of the choice of basis.*

PROOF: According to Theorem 2 for any vector f there is an expansion

$$Af = s_1(f, f_1)g_1 + s_2(f, f_2)g_2 + \cdots,$$

*Cf. Sec. 4.2.3.

where $Af_k = s_k g_k$, $A^* A f_k = \lambda_k f_k$, $\lambda_k = s_k^2 > 0$. Therefore

$$(A\varphi_k, \varphi_k) = \sum_{i=1}^{\infty} s_i (\varphi_k, f_i)(g_i, \varphi_k)$$

and

$$\sum_{k=1}^{\infty} |(A\varphi_k, \varphi_k)| \leqslant \sum_{i=1}^{\infty} s_i \left\{ \sum_{k=1}^{\infty} |(\varphi_k, f_i)|^2 \right\}^{1/2} \left\{ \sum_{k=1}^{\infty} |(\varphi_k, g_i)|^2 \right\}^{1/2} = \sum_{i=1}^{\infty} s_i.$$

Furthermore,

$$\sum_{k=1}^{\infty} (\varphi_k, f_i)(g_i, \varphi_k) = (g_i, f_i),$$

so that

$$\sum_{k=1}^{\infty} (A\varphi_k, \varphi_k) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} s_i (\varphi_k, f_i)(g_i, \varphi_k) = \sum_{i=1}^{\infty} s_i (g_i, f_i)$$

and the left-hand side is independent of the choice of the basis $\{\varphi_k\}_{k=1}^{\infty}$. ■

DEFINITION 3. The *spectral trace* of a nuclear operator A is the sum of its eigenvalues λ_k, i.e., the expression $\sum_{k=1}^{\infty} \lambda_k$.

It will be shown below that the spectral trace of a nuclear operator always exists and coincides with its matrix trace.

Let H be a complex Hilbert space. Then the following theorem holds.

THEOREM 4. *If A is a positive nuclear operator on H, then its matrix and spectral traces are finite and the equality* $\operatorname{Sp} A = \sum_{k=1}^{\infty} \lambda_k$ *holds, i.e., the matrix trace of the operator coincides with its spectral trace.*

PROOF: We remark that the operator A is positive and therefore, as shown above in Sec. 4.2.7, it is self-adjoint. Let $\{\lambda_k\}$ be the sequence of its positive eigenvalues, and $\{f_k\}$ an orthonormal sequence of eigenvectors. It is easy to verify that the positive square root of the operator A is defined for any vector $f \in H$ by the formula

$$A^{1/2} f = (A^{1/2})^* f = \sum_{k=1}^{\infty} \lambda_k^{1/2} (f, f_k) f_k, \quad \lambda_k^{1/2} > 0.$$

To prove this, since the positive square root is unique, it suffices to verify that $A^{1/2}$ is a positive operator and that $(A^{1/2})^2 = A$; but both of these assertions are obvious.

Furthermore, for any orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$

$$\sum_{k=1}^{\infty}(A\varphi_k, \varphi_k) = \sum_{k=1}^{\infty}((A^{1/2})^2\varphi_k, \varphi_k)$$
$$= \sum_{k=1}^{\infty}(A^{1/2}\varphi_k, A^{1/2}\varphi_k) == \sum_{k=1}^{\infty}\|A^{1/2}\varphi_k\|^2 < \infty.$$

Consequently the operator $A^{1/2}$ is an operator of Schmidt class. Therefore for any orthonormal basis $\{g_k\}_{k=1}^{\infty}$

$$\sum_{k=1}^{\infty}\|A^{1/2}g_k\|^2 = \sum_{k=1}^{\infty}\|A^{1/2}\varphi_k\|^2 = \sum_{k=1}^{\infty}\|A^{1/2}f_k\|^2 = \sum_{k=1}^{\infty}\lambda_k.$$

We remark that the sequence $\{f_k\}_{k=1}^{\infty}$ is not a basis of H; it is a basis only of the orthogonal complement of H_0, the nullspace of the operator A. However, if we choose an orthonormal basis $\{f'_k\}_{k=1}^{\infty}$ of H_0, we find that

$$\|A^{1/2}f'_k\|^2 = (A^{1/2}f'_k, A^{1/2}f'_k) = (Af'_k, f'_k) = 0,$$
$$\sum_{k=1}^{\infty}\|A^{1/2}\varphi_k\|^2 = \sum_{k=1}^{\infty}\|A^{1/2}f_k\|^2.$$

Therefore we could have written an equality above. Thus,

$$\sum_{k=1}^{\infty}(Ag_k, g_k) = \operatorname{Sp} A = \sum_{k=1}^{\infty}\lambda_k. \blacksquare$$

REMARK. It follows in particular from the proof of the theorem that if the series $\sum_{k=1}^{\infty}(A\varphi_k, \varphi_k)$ converges for every orthonormal basis $\{\varphi_k\}$, then the bounded linear operator A is nuclear.

Thus we have established trace theorems in two cases: for an operator in n-dimensional space, and for a positive nuclear operator on a separable complex Hilbert space H. The question naturally arises whether such theorems hold for arbitrary, not necessarily positive, nuclear operators. The answer to the question whether the matrix trace of an arbitrary nuclear operator equals its spectral trace will be given below. However the proof of this theorem requires significant advances in the study of nonself-adjoint completely continuous operators.

2.2. Properties of s-Numbers of Completely Continuous Operators

We recall that $R(A)$ always denotes the range of values of the operator A defined on H: $R(A) = \{Ax : x \in H\}$, and $Z(A)$ denotes its nullspace: $Z(A) = \{x : Ax = 0\}$.

LEMMA 1. *Let A be a continuous linear operator on the Hilbert space H. Then**

$$H = \overline{R(A^*)} \oplus Z(A), \quad H = \overline{R(A)} \oplus Z(A^*).$$

(The line denotes the closure of the corresponding linear manifolds; since the operators A and A^* are continuous, $Z(A)$ and $Z(A^*)$ are obviously always closed.)

PROOF: Let $h_0 \in H \ominus \overline{R(A^*)}$. Then $(h_0, A^*x) = 0 = (Ah_0, x)$ for any $x \in H$. Therefore $Ah_0 = 0$, i.e., $H \ominus \overline{R(A^*)} \subset Z(A)$. Conversely let $x_0 \in Z(A)$. Then $Ax_0 = 0$ and $0 = (Ax_0, x) = (x_0, A^*x)$ for all $x \in H$, i.e., $x_0 \perp R(A^*)$ and by the continuity of the inner product $x_0 \perp \overline{R(A^*)}$. But if a vector is orthogonal to $\overline{R(A^*)}$, then it belongs to $H \ominus \overline{R(A^*)}$. Consequently $Z(A) \subset H \ominus \overline{R(A^*)}$ and so $H = \overline{R(A^*)} \oplus Z(A)$. The proof of the second assertion is similar and was carried out in Proposition 6 of Sec. 4.2.13. ■

DEFINITION 4. An operator U mapping a Hilbert space H into itself is a *partial isometry* if it maps the subspace $H \ominus Z(U)$ isometrically onto $R(U)$.

It is easy to establish that *the range of values $R(U)$ of a partial isometry is closed.*

PROOF: In fact let $y_n \in R(U)$ and $y_n \to y$, $y \neq 0$. We shall show that there exists an element $x \in H \ominus Z(U)$ for which $Ux = y$.

Indeed, since $\{y_n\}$ is a convergent sequence, we have $\|y_n - y_m\| \to 0$ as $n, m \to \infty$ and $y_n \neq 0$ for $n > N$. Let x_n and x_m be elements in $H \ominus Z(U)$ such that $Ux_n = y_n$ and $Ux_m = y_m$. Then $\|y_n - y_m\| = \|Ux_n - Ux_m\| = \|x_n - x_m\| \to 0$. Since $H \ominus Z(U)$ is a subspace, the element $x = \lim_{n\to\infty} x_n$ belongs to $H \ominus Z(U)$. Further

$$\|Ux - Ux_n\| = \|x - x_n\| \to 0, \quad n \to \infty.$$

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*Cf. also Proposition 6 of Sec. 4.2.13.

Therefore $Ux_n = y_n \to Ux$ and since $y_n \to y$ as $n \to \infty$, we have $y = Ux$. If $y_n \to 0$, we have $U0 = 0$. ■

Taking account of the fact just proved and the assertion of Lemma 1, we conclude that a partial isometry U produces an isometric mapping of a subspace $R(U^*)$ onto $R(U)$ and U^* gives the inverse mapping of $R(U)$ onto $R(U^*)$.

LEMMA 2. *For any partial isometry U the equalities $U^*U = P_{R(U^*)}$ and $UU^* = P_{R(U)}$ hold, where $P_{R(U^*)}$ and $P_{R(U)}$ are the orthogonal projections on $R(U^*)$ and $R(U)$ respectively.*

PROOF: Indeed U^*U is a self-adjoint operator on H that vanishes on $Z(U)$. Therefore $(U^*Ux, y) = (x, U^*Uy) = 0$ for all $x \in H$ and $y \in Z(U)$. Hence $U^*Ux \in H \ominus Z(U)$ for all $x \in H$. Consequently for $x \in H \ominus Z(U)$ we also have $x - U^*Ux \in H \ominus Z(U)$. On the other hand

$$(x, y) = (Ux, Uy) = (U^*Ux, y), \quad (x - U^*Ux, y) = 0$$

for all $x \in H \ominus Z(U)$, $y \in H \ominus Z(U)$. Consequently the vector $x - U^*Ux$ is orthogonal to $H \ominus Z(U)$. But then $x = U^*Ux$, i.e., $U^*U = E$ on $H \ominus Z(U)$, and $U^*U = P_{H \ominus Z(U)} = P_{R(U^*)}$. The proof that $UU^* = P_{R(U)}$ is similar. ■

The polar representation of a complex number z in the form $z = re^{i\varphi}$ is well-known. Here $r > 0$, $r = |\bar{z}z|^{1/2}$, $\varphi = \arg z$. We remark that $|e^{i\varphi}| = 1$ and $e^{i\varphi} \cdot \overline{e^{i\varphi}} = 1$. An analogous representation holds for any continuous operator A.

LEMMA 3. *Let A be a continuous linear operator. Then there exist operators C and U with $C \geqslant 0$, C continuous, and U a partial isometry, such that $A = UC$.*

PROOF: Let $C = (A^*A)^{1/2}$, so that $C \geqslant 0$ and C is continuous. For any $x \in H$ we have

$$\|Cx\|^2 = (Cx, Cx) = (C^2x, x) = (A^*Ax, x) = (Ax, Ax) = \|Ax\|^2,$$

i.e., $\|Cx\| = \|Ax\|$ and therefore $Z(A) = Z(C)$, $\overline{R(A^*)} = \overline{R(C)}$. Furthermore if $y = Cx$, we set $Uy = Ax$; if $y \in Z(C)$, we set $Uy = 0$.

Thus we have defined an operator that maps $R(C)$ isometrically onto $R(A)$. We extend the operator U to all of $\overline{R(C)}$ by continuity. It is easy to see that

$$\overline{R(C)} = \overline{R(C^2)} = \overline{R(A^*A)} = \overline{R(A^*)} = R(U^*),$$

i.e., U is a partial isometry. We remark also that if $y = Cx_1 = Cx_2$, then $C(x_1 - x_2) = 0$. But then $A(x_1 - x_2) = 0$ also, i.e., $Ax_1 = Ax_2$, and so the operator U is unambiguously defined—the same element Uy corresponds to different representations of the element y in the form Ax. ■

REMARK. If the operator A is completely continuous, then $A = UC$, where $C \geqslant 0$ and C is a completely continuous operator.

In Sec. 4.2.9 we proved an eigenvector expansion theorem for a symmetric completely continuous operator A. It follows from this theorem that for any vector $f \in H$

$$(Af, f) = \sum_{i=1}^{\infty} \lambda_i |(f, \varphi_i)|^2,$$

where φ_i is an orthonormal system of eigenvectors of the operator A corresponding to nonzero eigenvalues λ_i. We now arrange the positive and negative eigenvalues into two sequences

$$\lambda_1^+ \geqslant \lambda_2^+ \geqslant \cdots, \quad \lambda_1^- \leqslant \lambda_2^- \leqslant \cdots,$$

where $\{\lambda_i^+\}$ are the positive eigenvalues and $\{\lambda_i^-\}$ are the negative eigenvalues. Let $\{\varphi_i^+\}$ and $\{\varphi_i^-\}$ be the eigenvectors corresponding to them. Then we can write

$$(Af, f) = \sum_i \lambda_i^+ |(f, \varphi_i^+)|^2 + \sum_i \lambda_i^- |(f, \varphi_i^-)|^2.$$

The following theorem is an immediate consequence of Theorem 10 of Sec. 4.2.10.

THEOREM 5 (Hilbert). *For any completely continuous symmetric linear operator A the following equalities hold:*

$$\lambda_1^+ = \max_{\|f\|=1} (Af, f) = (A\varphi_1^+, \varphi_1^+);$$

$$\lambda_{j+1}^+ = \max_{\substack{\|f\|=1, (f,\varphi_i)=0, \\ i=1,2,\dots,j,}} (Af, f) = (A\varphi_{j+1}^+, \varphi_{j+1}^+), \quad j = 1, 2, \dots .$$

Similarly,

$$\lambda_1^- = \min_{\|f\|=1} (Af, f) = (A\varphi_1^-, \varphi_1^-);$$

$$\lambda_{j+1}^- = \min_{\substack{\|f\|=1, (f,\varphi_i)=0, \\ i=1,2,\dots,j}} (Af, f) = (A\varphi_{j+1}^-, \varphi_{j+1}^-), \quad j = 1, 2, \dots .$$

We shall also prove the following theorem, which makes it possible to determine the nth eigenvalue.

THEOREM 6 (Courant). *Let $h_1, h_2, \dots, h_n$ be any n elements of H, let $V = V(h_1, h_2, \dots, h_n)$ be their linear span, and*

$$M = M(h_1, h_2, \dots, h_n) = \max_{\substack{\|f\|=1,\,(f,h_i)=0,\\ i=1,2,\dots,j}} (Af, f).$$

Then

$$\lambda_{j+1}^{+} = \min_{V \in R^j} M(h_1, h_2, \dots, h_j) = M(\varphi_1^{+}, \varphi_2^{+}, \dots, \varphi_j^{+}), \quad j = 1, 2, \dots,$$

where R^j is the set of all j-dimensional subspaces of H. Similarly

$$\lambda_{j+1}^{-} = \max_{V \in R^j} m(h_1, h_2, \dots, h_j) = m(\varphi_1^{-}, \varphi_2^{-}, \dots, \varphi_j^{-}),$$
$$m(h_1, h_2, \dots, h_j) = \min_{\substack{\|f\|=1,\,(f,h_i)=0,\\ i=1,2,\dots,j}} (Af, f), \quad j = 1, 2, \dots .$$

PROOF: We shall prove the assertion about λ_{j+1}^{+}. As already mentioned,

$$M(\varphi_1^{+}, \varphi_2^{+}, \dots, \varphi_j^{+}) = \lambda_{j+1}^{+}.$$

We shall show that for arbitrary vectors $h_1, h_2, \dots, h_j$ we have the inequality

$$M(h_1, h_2, \dots, h_j) \geqslant \lambda_{j+1}^{+}.$$

Consider the system

$$\sum_{i=1}^{j+1} f_i(\varphi_i^{+}, h_k) = 0, \quad k = 1, 2, \dots, j.$$

The system contains j linear homogeneous equations with $j+1$ unknowns. Such a system always has a nonzero solution $f_1^0, f_2^0, \dots, f_{j+1}^0$. Let $\sum_{i=1}^{j+1} |f_i^0|^2 = 1$. Then $f^0 = \sum_{i=1}^{j+1} f_i^0 \varphi_i^{+}$ is orthogonal to all the vectors h_k, $k = 1, 2, \dots, j$, $(f^0, h_k) = 0$ and $\|f^0\| = 1$ by Parseval's equality. Thus

$$(Af^0, f^0) \leqslant \max_{\|f\|=1,\,(f,h_i)=0} (Af, f) = M(h_1, h_2, \dots, h_j).$$

On the other hand

$$(Af^0, f^0) = \sum_{i,k=1}^{j+1} f_i^0 \bar{f}_k^0 (A\varphi_i^+, \varphi_k^+) = \sum_{i=1}^{j+1} \lambda_i^+ |f_i^0|^2 \geqslant \lambda_{j+1}^+ \sum_{i=1}^{j+1} |f_i^0|^2 = \lambda_{j+1}^+ .$$

Here we have used the fact that $\lambda_1^+ \geqslant \lambda_2^+ \geqslant \cdots \geqslant \lambda_j \geqslant \lambda_{j+1}^+$. ■

The theorem just proved makes it possible to compare the eigenvalues of different operators.

COROLLARY 1. *If $0 \leqslant A \leqslant B$ and A and B are completely continuous operators, then $\lambda_j(A) \leqslant \lambda_j(B)$, where $\lambda_j(A)$ is the jth eigenvalue of A and $\lambda_j(B)$ is the jth eigenvalue of B when the eigenvalues are arranged in decreasing order according to multiplicity.*

PROOF: Indeed in this case the nonzero eigenvalues of the operators A and B are positive. We obtain

$$\lambda_j(B) = \max_{\substack{\|f\|_1,\, f\perp\varphi_i, \\ i=1,2,\ldots,j-1}} (Bf, f) \geqslant \max_{\substack{\|f\|=1,\, f\perp\varphi_i, \\ i=1,2,\ldots,j-1}} (Af, f)$$
$$\geqslant \min_{V\in R^{j-1}} \max_{\substack{f\in V^\perp \\ \|f\|=1}} (Af, f) = \lambda_j(A),$$

where φ_i are the eigenvectors of the operator B:

$$B\varphi_i = \lambda_i(B)\varphi_i, \quad i = 1, 2, \ldots, j-1, \quad \text{and } V = V(h_1, h_2, \ldots, h_{j-1})$$

is the linear span of the vectors $h_1, h_2, \ldots, h_{j-1}$, R^{j-1} is the set of all $j-1$-dimensional subspaces of H, and $V^\perp$ is the orthogonal complement of V. ■

COROLLARY 2. *Let A be a completely continuous linear operator and B a bounded linear operator. Let $s_k(BA)$ be the s-numbers of the completely continuous operator BA and $s_k(A)$ the s-numbers of the operator A, both arranged in decreasing order according to multiplicity. Then*

$$s_k(BA) \leqslant \|B\| s_k(A).$$

PROOF: Indeed by the definition of s-numbers

$$s_k^2(BA) = \lambda_k((BA)^*(BA)), \quad s_k^2(A) = \lambda_k(A^*A).$$

On the other hand it is easy to see that

$$((BA)^*BAf, f) = \|BAf\|^2 \leqslant \|B\|^2\|Af\|^2$$
$$= \|B\|^2(Af, Af) = (\|B\|^2A^*Af, f),$$

i.e.,

$$0 \leqslant (BA)^*BA \leqslant \|B\|^2A^*A.$$

Applying the result obtained in Corollary 1, we arrive at the required result. ■

LEMMA 4. *Let A be a completely continuous operator. Then*

$$s_k(A) = s_k(A^*).$$

PROOF: According to what was proved above (cf. Theorem 2) we can write the expansion $A\varphi = \sum\limits_{k=1}^{\infty} s_k(A)(\varphi, \varphi_k)\psi_k$, for the completely continuous operator A, where φ_k are orthonormal eigenvectors of the operator A^*A and ψ_k is some other orthonormal system. We shall now find the form of the operator adjoint to A. We have

$$(A\varphi, \psi) = \sum_{k=1}^{\infty} s_k(A)(\varphi, \varphi_k)(\psi_k, \psi) = \Big(\varphi, \sum_{k=1}^{\infty} s_k(A)(\psi, \psi_k)\varphi_k\Big) = (\varphi, A^*\psi).$$

Thus

$$A^*\psi = \sum_{k=1}^{\infty} s_k(A)(\psi, \psi_k)\varphi_k,$$

and it is clear from the expressions for A and A^* that

$$A\varphi_k = s_k(A)\psi_k, \qquad A^*\psi_k = s_k(A)\varphi_k,$$
$$A^*A\varphi_k = s_k^2(A)\varphi_k, \quad AA^*\psi_k = s_k^2(A)\psi_k.$$

From this we find by the definition of the s-numbers that

$$s_k(A) = s_k(A^*). \blacksquare$$

COROLLARY. *Let A be a completely continuous operator and B a bounded linear operator. Then*

$$s_k(AB) \leqslant \|B\| s_k(A).$$

PROOF: Since $\|B\| = \|B^*\|$, it follows from the lemma and Corollary 2 of Theorem 6 that

$$s_k(AB) = s_k(B^* A^*) \leqslant s_k(A^*)\|B^*\| = \|B\| s_k(A). \blacksquare$$

We shall now prove a formula that we shall make use of below in studying the properties of the s-numbers of completely continuous operators. This formula is known as the *Binet-Cauchy formula.*

LEMMA 5. *Let C be a square $m \times m$ matrix, B a rectangular $n \times m$ matrix with $n > m$, A a rectangular $m \times n$ matrix, and $C = AB$. Then*

$$\det C = \sum_{1 \leqslant s_1 < \cdots < s_m \leqslant n \leqslant \infty} \begin{vmatrix} a_{1s_1} & \dots & a_{1s_m} \\ a_{ms_1} & \dots & a_{ms_m} \end{vmatrix} \cdot \begin{vmatrix} b_{s_1 1} & \dots & b_{s_1 m} \\ b_{s_m 1} & \dots & b_{s_m m} \end{vmatrix}.$$

PROOF: We write an expression for the general element of the matrix C

$$c_{ij} = \sum_{s=1}^{n} a_{is} b_{sj}.$$

Then

$$\det C = \det \left\| \begin{matrix} \sum\limits_{s_1=1}^{n} a_{1s_1} b_{s_1 1} & \dots & \sum\limits_{s_m=1}^{n} a_{1s_m} b_{s_m m} \\ \sum\limits_{s_1=1}^{n} a_{ms_1} b_{s_1 1} & \dots & \sum\limits_{s_m=1}^{n} a_{ms_m} b_{s_m m} \end{matrix} \right\|.$$

Using an elementary property of determinants, we write

$$\det C = \sum_{s_1, \dots, s_m = 1}^{n} \begin{vmatrix} a_{1s_1} & \dots & a_{1s_m} \\ a_{ms_1} & \dots & a_{ms_m} \end{vmatrix} b_{s_1 1} b_{s_2 2} \cdots b_{s_m m}.$$

The numbers $s_1, s_2, \dots, s_m$ vary independently of one another. We shall suppose that they are all different, since otherwise

$$\begin{vmatrix} a_{1s_1} & \dots & a_{1s_m} \\ a_{ms_1} & \dots & a_{ms_m} \end{vmatrix} = 0.$$

Some of the terms in the sum differ only in the order of columns in the determinant. Combining these into a single term, we can make use of properties of permutations and determinants to write

$$\det C = \sum_{1\leqslant s_1<s_2<\cdots<s_m\leqslant n\leqslant\infty} \begin{vmatrix} a_{1s_1} & \dots & a_{1s_m} \\ a_{ms_1} & \dots & a_{ms_m} \end{vmatrix} \cdot \begin{vmatrix} b_{s_1 1} & \dots & b_{s^1 m} \\ b_{s_m 1} & \dots & b_{s_m m} \end{vmatrix},$$

which was to be proved. ■

LEMMA 6. *Let A be a completely continuous operator and consider an arbitrary set of vectors $h_1, h_2, \dots, h_m$. Then*

$$\det \|(Ah_j, Ah_k)\|_{j,k=1}^m \leqslant s_1^2(A)\cdots s_m^2(A) \det \|(h_j, h_k)\|_{j,k=1}^m .$$

PROOF: Let $\{\varphi_i\}$ be a complete orthonormal system of eigenvectors of the operator C given by $C = (A^* A)^{1/2}$. Then

$$\lambda_n(A^* A) = s_n^2(A), \quad (Ah_j, Ah_k) = (A^* Ah_j, h_k).$$

Let

$$h_j = \sum_{n=1}^{\infty}(h_j, \varphi_n)\varphi_n, \quad h_k = \sum_{p=1}^{\infty}(h_k, \varphi_p)\varphi_p.$$

We write

$$(A^* Ah_j, h_k) = \Big(\sum_{n=1}^{\infty} s_n^2(A)(h_j, \varphi_n)\varphi_n, \sum_{p=1}^{\infty}(h_k, \varphi_p)\varphi_p\Big)$$

$$= \sum_{n=1}^{\infty} s_n^2(A)(h_j, \varphi_n)(\varphi_n, h_k).$$

We introduce the matrix $B = \|b_{ij}\|$, $b_{ij} = s_j \cdot (h_i, \varphi_j)$. Then $B^* = \|b_{ij}^*\|$, where $b_{ij}^* = s_i(\varphi_i, h_j)$, since $\overline{(h_i, \varphi_j)} = (\varphi_j, h_i)$. We denote the matrix $\|(Ah_j, Ah_k)\|_{j,k=1}^m$ by K. Then $K = BB^*$. We apply the formula obtained in the preceding lemma to the determinant of the matrix K. We have

$$\det K = \sum_{1\leqslant k_1<\cdots<k_m\leqslant\infty} \begin{vmatrix} s_{k_1}\cdot(h_1, \varphi_{k_1}) & \dots & s_{k_m}\cdot(h_1, \varphi_{k_m}) \\ s_{k_1}\cdot(h_m, \varphi_{k_1}) & \dots & s_{k_m}\cdot(h_m, \varphi_{k_m}) \end{vmatrix} \times$$

$$\times \begin{vmatrix} s_{k_1}\cdot(\varphi_{k_1}, h_1) & \dots & s_{k_1}\cdot(\varphi_{k_1}, h_m) \\ s_{k_m}\cdot(\varphi_{k_m}, h_1) & \dots & s_{k_m}\cdot(\varphi_{k_m}, h_m) \end{vmatrix}$$

$$= \sum_{1\leqslant k_1<\cdots<k_m\leqslant\infty} s_{k_1}^2\cdots s_{k_m}^2 \det\|(h_i, \varphi_k)\| \cdot \det\|\overline{(h_j, \varphi_k)}\|,$$

where $\det \|(h_i, \varphi_k)\|$ and $\det \|\overline{(h_j, \varphi_k)}\|$ denote the determinants occurring above with the numbers s_{k_j} factored out.

The numbers k_j are distinct and arranged in increasing order. Then the numbers s_{k_j} will be arranged in decreasing order and

$$s_1 \geqslant s_{k_1}, \ldots, s_m \geqslant s_{k_m}.$$

Therefore if we make use of the Binet-Cauchy formula again, we obtain

$$\det K \leqslant s_1^2 \cdots s_m^2 \cdot \sum_{1 \leqslant k_1 < \cdots < k_m} \det \|(h_i, \varphi_k)\| \cdot \det \|\overline{(h_j, \varphi_k)}\|$$
$$= s_1^2 \cdots s_m^2 \det \|(h_i, h_j)\|_{i,j=1}^m,$$

since $\sum_{k=1}^{\infty} (h_i, \varphi_k)\overline{(h_j, \varphi_k)} = (h_i, h_j)$, which was to be proved. ■

2.3. Estimates for the Eigenvalues of a Completely Continuous Operator

The following proposition, which is of great importance, makes it possible to estimate the product of the eigenvalues of a completely continuous operator in terms of the product of the s-numbers of the same operator.

THEOREM 7. (H. Weyl). *Let A be a completely continuous operator and $\lambda_1(A), \lambda_2(A), \ldots, \lambda_k(A)$ its eigenvalues, $k \leqslant \nu(A)$, $\nu(A) = \sum_{\lambda_i \neq 0} \dim L_{\lambda_i}$, where L_{λ_i} are the root subspaces of the operator A corresponding to the eigenvalues λ_i. Then*

$$|\lambda_1(A)\lambda_2(A), \cdots \lambda_k(A)| \leqslant s_1(A)s_2(A) \cdots s_k(A).$$

PROOF: Let $L = \bigcup_{\lambda_i \neq 0} L_{\lambda_i}$. We choose a Schur basis $\{\varphi_i\}$ in L (cf. Sec. 4.3.3). Then

$$A\varphi_j = \alpha_{j1}\varphi_1 + \cdots + \alpha_{jj}\varphi_j, \quad \alpha_{ij} = (A\varphi_i, \varphi_i), \quad \alpha_{ii} = \lambda_i = (A\varphi_i, \varphi_i).$$

We have

$$\det \|(A\varphi_i, A\varphi_j)\|_{i,j=1}^k \leqslant s_1^2(A)s_2^2(A) \cdots s_k^2(A) \det \|(\varphi_i, \varphi_j)\|_{i,j=1}^k.$$

But

$$(A\varphi_i, A\varphi_j) = \sum_{p=1}^{\min(i,j)} (A\varphi_i, \varphi_p)\overline{(A\varphi_j, \varphi_p)}.$$

Then

$$\det \|(A\varphi_i A\varphi_j)\| = \det \|(A\varphi_i, \varphi_j)\| \cdot \det \|\overline{(A\varphi_i, \varphi_j)}\|$$
$$= \det \|(A\varphi_i, \varphi_j)\| \cdot \overline{\det \|(A\varphi_i, \varphi_j)\|} = \big| \det \|(A\varphi_i, \varphi_j)\| \big|^2.$$

Taking account of the fact that

$$\det \|A\varphi_i, \varphi_j\|_{i,j=1}^k = \lambda_1(A)\lambda_2(A)\cdots\lambda_k(A),$$

we find that

$$\det \|(A\varphi_i, A\varphi_j)\| = |\lambda_1(A)|^2 |\lambda_2(A)|^2 \cdots |\lambda_k(A)|^2.$$

We write finally:

$$\det \|(A\varphi_i, A\varphi_j)\|_{i,j=1}^k = |\lambda_1(A) \cdot \lambda_2(A) \cdots \lambda_k(A)|^2 \leqslant s_1^2(A) s_2^2(A) \cdots s_k^2(A),$$

since $\det \|(\varphi_i, \varphi_j)\|_{i,j=1}^k = 1$, which was to be proved. ■

LEMMA 7. *Let $\Phi(x)$ $(-\infty \leqslant x < \infty)$ be a convex function such that $\Phi(-\infty) = \lim\limits_{x \to -\infty} \Phi(x) = 0$, and let $\{a_j\}$, $\{b_j\}$ be nonincreasing sequences of real numbers such that*

$$\sum_{j=1}^k a_j \leqslant \sum_{j=1}^k b_j, \quad k = 1, 2, \dots .$$

Then

$$\sum_{j=1}^k \Phi(a_j) \leqslant \sum_{j=1}^k \Phi(b_j), \quad k = 1, 2, \dots .$$

PROOF: Consider first the function $\Phi(t) = e^t$. For an arbitrary function we can then carry out the necessary changes. We denote by y_+ the function

$$y_+ = \max(y, 0).$$

Then

$$\int_{-\infty}^{+\infty} (t-s)_+ e^s \, ds = \int_{-\infty}^{t} (t-s) e^s \, ds$$
$$= \int_{-\infty}^{t} t e^s \, ds - \int_{-\infty}^{t} s e^s \, ds = t e^s \big|_{-\infty}^{t} - s e^s \big|_{-\infty}^{t} + e^s \big|_{-\infty}^{t} = e^t.$$

Thus

$$e^t = \int_{-\infty}^{+\infty} (t-s)_+ \, d(e^s)'.$$

For an arbitrary function $\Phi(t)$ satisfying the hypotheses of the lemma we have the representation

$$\Phi(t) = \int_{-\infty}^{+\infty} (t-s)_+ \, d\Phi'(s).$$

Indeed

$$\begin{aligned}\int_{-N}^{\infty} (t-s)_+ \, d\Phi'(s) &= \int_{-N}^{t} (t-s) \, d\Phi'(s) \\ &= \int_{-N}^{t} \Phi'(s) \, ds - (t+N)\Phi'(-N), \quad N > 0.\end{aligned}$$

The left-hand side of the equality just written is positive, and so

$$(t+N)\Phi'(-N) \leqslant \int_{-N}^{t} \Phi'(s) \, ds = \Phi(t) - \Phi(-N) \leqslant \Phi(t), \quad t > -N.$$

Consequently

$$\lim_{N\to\infty} \Phi'(-N) = 0, \quad \overline{\lim_{N\to\infty}} \, N\Phi'(-N) < \infty.$$

Since $\Phi(t) \to 0$ as $t \to -\infty$, we have

$$\lim_{N\to\infty} (t+N)\Phi'(-N) = \lim_{N\to\infty} N\Phi'(-N) = 0.$$

Finally, in the equality

$$\int_{-N}^{\infty} (t-s)_+ \, d\Phi'(s) = \int_{-N}^{t} \Phi'(s) \, ds - (t+N)\Phi'(-N)$$

we need only pass to the limit as $N \to \infty$, and the required representation for $\Phi(t)$ will be obtained.

It follows from this representation that

$$\sum_{j=1}^{k} \Phi(a_j) = \int_{-\infty}^{+\infty} A_k(s) \, d\Phi'(s),$$

where

$$A_k(s) = \sum_{j=1}^{k} (a_j - s)_+ .$$

Analogously

$$\sum_{j=1}^{k} \Phi(b_j) = \int_{-\infty}^{+\infty} B_k(s)\, d\Phi'(s),$$

where

$$B_k(s) = \sum_{j=1}^{k} (b_j - s)_+ .$$

(We remark that we have denoted the left-hand derivative of the convex function $\Phi(s)$, which is known to exist everywhere and to be a nonnegative nondecreasing function, by the symbol $\Phi'(s)$). We shall now prove that the functions $A_k(s)$ and $B_k(s)$ are connected by the relation

$$A_k(s) \leqslant B_k(s), \quad k = 1, 2, \dots .$$

We consider several cases.

A. Let $s \geqslant b_1$. Then

$$A_k(s) = B_k(s) \equiv 0,$$

and the inequality is satisfied trivially.

B. Let $s \leqslant \min(a_k, b_k)$. Then

$$A_k(s) = \sum_{j=1}^{k} a_j - ks, \quad B_k(s) = \sum_{j=1}^{k} b_j - ks,$$

i.e.,

$$A_k(s) \leqslant B_k(s).$$

C. Consider the last case

$$a_{q+1} \leqslant s < a_q, \quad b_{p+1} \leqslant s < b_p \quad (p, q \leqslant k).$$

Then for $p \geqslant q$ we have

$$A_k(s) = \sum_{j=1}^{q} a_j - qs \leqslant \sum_{j=1}^{q} b_j - qs + (b_{q+1} - s) + \cdots + (b_p - s) = B_k(s),$$

since

$$\sum_{j=1}^{q} a_j \leqslant \sum_{j=1}^{q} b_j, \quad b_{q+1} \geqslant s, \dots b_p \geqslant s.$$

For $p < q$

$$A_k(s) = \sum_{j=1}^{q} a_j - qs \leqslant \sum_{j=1}^{q} b_j - qs = \sum_{j=1}^{p} b_j - ps$$
$$+ (b_{p+1} - s) + \cdots + (b_q - s) \leqslant \sum_{j=1}^{p} b_j - ps = B_k(s),$$

since

$$b_{p+1} \leqslant s, \dots, b_q \leqslant s.$$

The lemma is thus proved. ∎

As a corollary of this lemma we obtain new inequalities for the eigenvalues and s-numbers of a completely continuous operator.

THEOREM 8 (The H. Weyl majorant theorem). *Let the function $f(x)$ defined on $[0, \infty)$ satisfy $f(0) = 0$, and let it be such that $f(e^t)$ is convex. Then for any completely continuous operator A and any $k \leqslant \nu(A) = \sum_{\lambda_i \neq 0} \dim L_{\lambda_i}$ the following inequalities hold:*

$$\sum_{j=1}^{k} f(|\lambda_i(A)|) \leqslant \sum_{j=1}^{k} f(s_j(A)), \quad \lambda_j \neq 0, \quad s_j \neq 0.$$

PROOF: Consider the function $\Phi = f(e^t)$ and the sequences $a_j = \ln|\lambda_j(A)|$, $b_j = \ln(s_j(A))$. Applying Theorem 7 and Lemma 7, we obtain the inequalities

$$\sum_{j=1}^{k} f(|\lambda_j(A)|) \leqslant \sum_{j=1}^{k} f(s_j(A)). \blacksquare$$

COROLLARY 1. *For any completely continuous operator A the inequalities*

$$\sum_{j=1}^{k} |\lambda_j(A)| \leqslant \sum_{j=1}^{k} s_j(A)$$

hold for $k \leqslant \nu(A)$.

Indeed the function $f(x) = x$ satisfies the hypotheses of the preceding theorem.

COROLLARY 2. *Let A be a completely continuous operator and $\lambda > 0$. Then*

$$\prod_{j=1}^{k}(1+\lambda|\lambda_j(A)|) \leqslant \prod_{j=1}^{k}(1+\lambda s_j(A)), \quad k \leqslant \nu(A).$$

To prove this it suffices to consider the function $f(x) = \ln(1+\lambda x)$.

REMARK. In the preceding lemmas and theorems we encountered the sequences $\{\lambda_j(A)\}$ and $\{s_j(A)\}$—the eigenvalues and s-numbers respectively of the completely continuous operator A. In the case of a completely continuous operator the s-numbers in general form an infinite sequence. The eigenvalues λ_i are enumerated according to their algebraic multiplicity (so that each number λ_j occurs in the sums a number of times equal to its multiplicity) and in decreasing order of their absolute values. The sums and products used above may also contain an infinite number of terms and factors and may be divergent.

We now resume the study of the s-numbers of completely continuous operators.

LEMMA 8 (Ky Fan). *Let A be a completely continuous operator and $\{\varphi_i\}$ an orthonormal system in H. Let U be a unitary operator. Then*

$$\sup_{U\{\varphi_i\}}\left|\sum_{i=1}^{n}(UA\varphi_i,\varphi_i)\right| = \sum_{i=1}^{n} s_i(A), \quad n = 1,2,\ldots .$$

PROOF: Let P_n be the orthogonal projection on a subspace L with basis $\varphi_1,\varphi_2,\ldots,\varphi_n$. Then

$$(UAP_n\varphi_i, P_n\varphi_i) = (UA\varphi_i,\varphi_i) = (P_nUAP_n\varphi_i,\varphi_i).$$

We introduce the notation

$$\tilde{A} = P_nUAP_n.$$

Then

$$\sum_{i=1}^{n}(\tilde{A}\varphi_i,\varphi_i) = \sum_{i=1}^{n}\lambda_i(\tilde{A}),$$

where $\lambda_i(\tilde{A})$ are the eigenvalues of the matrix $\tilde{A}$ defined by the operator $\tilde{A}$ on V. Using Theorem 1 and Corollary 1 to Theorem 8, we can write

$$\operatorname{Sp}\tilde{A} = \sum_{i=1}^{n}(\tilde{A}\varphi_i, \varphi_i) = \sum_{i=1}^{n}\lambda_i(\tilde{A}),$$

$$\left|\sum_{i=1}^{n}(\tilde{A}\varphi_i, \varphi_i)\right| \leqslant \sum_{i=1}^{n}|\lambda_i(\tilde{A})| \leqslant \sum_{i=1}^{n} s_i(\tilde{A}) \leqslant s_i(A),$$

$$\left|\sum_{i=1}^{n}(UA\varphi_i, \varphi_i)\right| \leqslant \sum_{i=1}^{n} s_i(A),$$

since

$$s_i(\tilde{A}) = s_i(P_nUAP_n) \leqslant s_i(P_nUA)\|P_n\| \leqslant \|P_n\|s_i(UA) \leqslant s_i(A)\|U\| = s_i(A).$$

We shall prove that equality is attained in this inequality. Let

$$A = UC$$

be the decomposition of the operator A (cf. Lemma 3 above) and φ_i^0 the orthonormalized eigenvectors of the operator C:

$$C\varphi_i^0 = s_i\varphi_i^0.$$

We choose a unitary operator U_0 which acts as follows on φ_i^0, $i = 1, 2, \ldots, n$:

$$U_0UC\varphi_i^0 = s_i\varphi_i^0, \quad \|U_0UC\varphi_i^0\| = s_i.$$

Since

$$\|UC\varphi_i^0\| = s_i\|U\varphi_i^0\| = s_i\|\varphi_i^0\| = s_i,$$

it follows that

$$\|U_0UC\varphi_i^0\| = \|UC\varphi_i^0\|.$$

We define U_0 on all of H so that it remains a unitary operator. For this operator we shall have

$$\sum_{i=1}^{n}(U_0UC\varphi_i^0, \varphi_i^0) = \sum_{i=1}^{n}(U_0A\varphi_i^0, \varphi_i^0) = \sum_{i=1}^{n} s_i(A). \blacksquare$$

COROLLARY. *Let A be a completely continuous operator. Then for any orthonormal system of vectors $\{\varphi_j\}$ we have the inequality*

$$\sum_{i=1}^{n} |(A\varphi_i, \varphi_i)| \leqslant \sum_{i=1}^{n} s_i(A), \quad n = 1, 2, \dots .$$

PROOF: Indeed

$$\left|\sum_{i=1}^{n} (UA\varphi_i, \varphi_i)\right| \leqslant \sum_{i=1}^{n} |(UA\varphi_i \varphi_i)| \leqslant \sum_{i=1}^{n} s_i(A).$$

We choose U so that if

$$(A\varphi_k, \varphi_k) = |(A\varphi_k, \varphi_k)| e^{i\theta_k}, \quad \theta_k = \arg(A\varphi_k, \varphi_k),$$

then

$$U^* \varphi_k = e^{i\theta_k} \varphi_k.$$

Then

$$(UA\varphi_k, \varphi_k) = (A\varphi_k, U^* \varphi_k) = e^{-i\theta_k} (A\varphi_k, \varphi_k) = |(A\varphi_k, \varphi_k)|,$$

so that

$$\sum_{i=1}^{n} |(A\varphi_i, \varphi_i)| \leqslant s_i(A). \blacksquare$$

2.4. Estimates for s-Numbers of Products and Sums of Completely Continuous Linear Operators

Various estimates have been obtained above for the eigenvalues and s-numbers of a completely continuous operator and for certain functions of the eigenvalues and s-numbers of a completely continuous operator. The estimation technique that was developed makes it possible to obtain analogous results for the eigenvalues and s-numbers of a sum or composition of two (or more) completely continuous operators. We recall that throughout we are considering only linear operators.

LEMMA 9. *For any two completely continuous operators A and B the following relations hold:*

$$\prod_{j=1}^{n} s_j(AB) \leqslant \prod_{j=1}^{n} s_j(A) \prod_{j=1}^{n} s_j(B), \quad n = 1, 2, \dots,$$

$$\sum_{j=1}^{n} s_j(A+B) \leqslant \sum_{j=1}^{n} s_j(A) + \sum_{j=1}^{n} s_j(B), \quad n = 1, 2, \dots .$$

PROOF: According to Lemma 6 for an arbitrary complete orthonormal system of vectors $\{\varphi_j\}$ we have the equality

$$\det \|(AB\varphi_i, AB\varphi_j)\|_{i,j=1}^n$$

$$\leqslant \prod_{j=1}^n s_j^2(A) \det \|(B\varphi_i, B\varphi_j)\|_{i,j=1}^n \leqslant \prod_{j=1}^n s_j^2(A) \prod_{j=1}^n s_j^2(B).$$

As $\{\varphi_j\}$ we choose a system of unit eigenvectors of the self-adjoint operator $B^* A^* AB$. Then we have the equality

$$\det \|(AB\varphi_i, AB\varphi_j)\|_{i,j=1}^n = \prod_{j=1}^n s_j^2(AB),$$

and the first inequality is proved.

According to Lemma 8 there exist an orthonormal system of vectors $\{\varphi_j\}$ and a unitary operator U such that

$$\Big|\sum_{j=1}^n (U(A+B)\varphi_j, \varphi_j)\Big| = \sum_{j=1}^n s_j(A+B).$$

Again applying this lemma, we write

$$\sum_{j=1}^n s_i(A+B) \leqslant \Big|\sum_{j=1}^n (UA\varphi_j, \varphi_j)\Big| + \Big|\sum_{j=1}^n (UB\varphi_j, \varphi_j)\Big|$$

$$\leqslant \sum_{j=1}^n s_i(A) + \sum_{j=1}^n s_i(B),$$

which was to be proved. ■

THEOREM 9 (Ky Fan). *Let A and B be completely continuous operators and let the function $f(x)$ $(0 \leqslant x < \infty)$ be a nondecreasing convex function with $f(0) = 0$. Then*

$$\sum_{j=1}^\infty f(s_j(A+B)) \leqslant \sum_{j=1}^\infty f(s_j(A) + s_j(B)).$$

PROOF: Indeed, applying Lemma 9 and Lemma 7 with $a_j = s_j(A+B)$, $b_j = s_j(A) + s_j(B)$, and

$$\Phi(x) = \begin{cases} f(x), & 0 \leqslant x < \infty, \\ 0, & -\infty \leqslant x < 0, \end{cases}$$

we obtain the required inequality on passing to the limit as the upper index n of the sums tends to infinity. ■

THEOREM 10 (A. Horn). *Let the function $f(x)$ $(0 \leqslant x < \infty,\ f(0) = 0)$, be such that $f(e^t)$ $(-\infty \leqslant t < \infty)$ is convex. Let A and B be two arbitrary completely continuous operators. Then*

$$\sum_{j=1}^{\infty} f(s_j(AB)) \leqslant \sum_{j=1}^{\infty} f(s_j(A) \cdot s_j(B)).$$

PROOF: Just as in the proof of Theorem 9, it suffices to apply Lemmas 7 and 9 with

$$a_j = s_j(AB), \quad b_j = s_j(A)s_j(B),$$

and the proof is finished. ■

COROLLARY. *For any two completely continuous operators A and B we have the relations*

$$\sum_{j=1}^{n} s_j(AB) \leqslant \sum_{j=1}^{n} s_j(A) \cdot s_j(B), \quad n = 1, 2, \ldots .$$

Indeed, one has only to set $f(x) = x$ in Theorem 10 and apply the theorem just proved.

2.5. The Trace Theorem for a Nuclear Operator

We now return to the study of nuclear operators. It was shown at the beginning of the present section that if A is a nuclear operator, then for any choice of an orthonormal basis $\{\varphi_i\}$ in H the series $\sum\limits_{i=1}^{\infty}(A\varphi_i, \varphi_i)$ converges absolutely and its sum is independent of the choice of basis. The following converse of this assertion is true.

THEOREM 11. *Let A be a bounded linear operator defined on all of H, and suppose that for any orthonormal basis $\{\varphi_i\}$ the series $\operatorname{Sp} A = \sum\limits_{i=1}^{\infty}(A\varphi_i, \varphi_i)$ converges absolutely. Then A is a nuclear operator.*

PROOF: Suppose first that A is a self-adjoint bounded operator and $E(\lambda)$ is its spectral family. It follows from the spectral theorem that the subspaces $E(0)H = H_-$ and $(E - E(0))H = H_+$ are invariant under the operator A, and both the operators $A_- = -AE(0)$ and $A_+ = A(E - E(0))$ are nonnegative. Let $\{f_j\}$ and $\{\psi_j\}$ be orthonormal bases of H_+ and H_- respectively. It follows from the hypothesis of the theorem that

$$\sum_{j=1}^{\infty}(A_+ f_j, f_j) < \infty, \quad \sum_{j=1}^{\infty}(A_- \psi_j, \psi_j) < \infty.$$

It follows by Theorem 4* that A_+ and A_- are nuclear operators. Since A is the difference of nuclear operators, it is itself nuclear. Indeed it follows from Lemma 9 that the series of s-numbers of the operator that is the sum (or difference) of two operators whose series of s-numbers converge is a convergent series.

Now consider the general case of a nonself-adjoint operator A with a finite matrix trace. The adjoint operator A^* obviously has a finite matrix trace also. Therefore the operators

$$A_R = \frac{1}{2}(A + A^*) \quad A_I = \frac{1}{2i}(A - A^*)$$

have finite matrix traces. These operators are obviously also self-adjoint. From what has been shown they are also nuclear. Just as above we conclude that the operator

$$A = A_R + iA_I$$

is nuclear. ■

We conclude our study of nuclear operators with a theorem on the equality of the matrix trace and the spectral trace of such an operator.

THEOREM 12 (Lidskii). *If the operator A is nuclear, then its matrix trace coincides with its spectral trace:*

$$\sum_{j=1}^{\infty}(A\varphi_j, \varphi_j) = \sum_i \lambda_i(A),$$

where $\{\varphi_i\}$ is an arbitrary orthonormal basis in H and $\lambda_i(A)$ are the eigenvalues of the operator A.

*Cf. the remark after Theorem 4.

PROOF: We first prove the thoerem in the case when the operator A is a Volterra operator. In this case there are no nonzero eigenvalues. Hence we must have

$$\operatorname{Sp}(A) = 0.$$

Let P_n be a finite-dimensional orthogonal projection for $n = 1, 2, \dots$ chosen in the following special manner: If $A = \sum_k s_k(\cdot, \varphi_k)\psi_k$, where $\{\varphi_k\}$ and $\{\psi_k\}$ are two orthonormal systems and $A = UC$, $C\varphi_k = s_k\varphi_k$, then

$$P_n H = \{\varphi_1, \dots, \varphi_n\} = V(\varphi_1, \varphi_2, \dots, \varphi_n)$$

where V is the subspace spanned by $\varphi_1, \varphi_2, \dots, \varphi_n$. It is obvious that

$$AP_n f = \sum_k s_k(P_n f, \varphi_k)\psi_k = \sum_{k=1}^{n} s_k(f, \varphi_k)\psi_k.$$

Then

$$\|(A - AP_n)f\|^2 = \Big\| \sum_{k=n+1}^{\infty} s_k(f, \varphi_k)\psi_k \Big\|^2 = \sum_{k=n+1}^{\infty} s_k^2 |(f, \varphi_k)|^2 \leqslant s_{n+1}^2 \|f\|^2.$$

Consequently

$$\|A - AP_n\| \leqslant s_{n+1} \to 0.$$

If the operator A is a Volterra operator, it is obvious that $\lambda_k(AP_n) \to 0$ as $n \to \infty$. Indeed the operator AP_n is finite-dimensional and $AP_n - A = B_n$ has the property that $\|B_n\| \to 0$ as $n \to \infty$. The operator $(A - \lambda E)^{-1}$ exists if $\lambda \neq 0$. Then it is easy to see that

$$(AP_n - \lambda E)^{-1}$$

exists outside a disk O_n whose radius depends on n and tends to zero as $n \to \infty$:

$$\begin{aligned}(AP_n - \lambda E)^{-1} &= (AP_n - A + A - \lambda E)^{-1} \\ &= (A - \lambda E)^{-1}(E + B_n(A - \lambda E)^{-1})^{-1}.\end{aligned}$$

Let $\lambda_j^{(n)} = \lambda_j(AP_n)$ be the eigenvalues of the operator AP_n, $j = 1, 2, \dots, n$, enumerated according to multiplicity, with

$$|\lambda_1^{(n)}| \geqslant |\lambda_2^{(n)}| \geqslant \cdots.$$

Consider the functions

$$\Delta_n(\lambda) = \prod_{j=1}^{n} (1 - \lambda\lambda_j^{(n)}).$$

Then

$$\frac{\Delta_n'(\lambda)}{\Delta_n(\lambda)} = -\sum_{j=1}^{n} \frac{\lambda_j^{(n)}}{1 - \lambda\lambda_j^{(n)}}.$$

Let $|\lambda\lambda_1^{(n)}| < 1$. Expanding the fraction into a geometrical series, we obtain

$$\frac{\Delta_n'(\lambda)}{\Delta_n(\lambda)} = -\sum_{j=1}^{n}\sum_{k=0}^{\infty} \lambda_j^{(n)} [\lambda_i^{(n)}]^k \lambda^k = -\sum_{k=1}^{\infty} M_k^{(n)} \lambda^{k-1},$$

$$M_k^{(n)} = \mathrm{Sp}\,[(AP_n)]^k = \sum_{m=1}^{n} [\lambda_m^{(n)}]^k.$$

Furthermore

$$\sum_{k=1}^{n} [\lambda_m^{(n)}]^k \leqslant \sum_{m=1}^{n} |\lambda_m^{(n)}|^k \leqslant |\lambda_1^{(n)}|^{k-1} \sum_{m=1}^{n} |\lambda_m^{(n)}|.$$

By the Weyl majorant theorem

$$|\lambda_m^{(n)}| \leqslant s_m^{(n)} = s_m(AP_n).$$

The norms of the operators P_n are all equal to 1. Therefore

$$s_m(AP_n) \leqslant s_m(A).$$

Taking account of the fact that the operator A is nuclear, we conclude:

$$\sum_{m=1}^{n} [\lambda_m^{(n)}]^k \leqslant |\lambda_1^{(n)}|^{k-1} \sum_{m=1}^{n} |\lambda_m^{(n)}| \leqslant |\lambda_1^{(n)}|^{k-1} \sum_{m=1}^{n} s_m(A)$$

$$\leqslant |\lambda_1^{(n)}|^{k-1} \sum_{k=1}^{\infty} s_k(A) = |\lambda_1^{(n)}|^{k-1} C_0, \quad \text{where } C_0 = \sum_{k=1}^{\infty} s_k(A) < \infty.$$

Thus

$$\left|\frac{\Delta_n'(\lambda)}{\Delta_n(\lambda)} + \mathrm{Sp}\,A\right| \leqslant |M_1^{(n)} - \mathrm{Sp}\,A| + \left|\sum_{k=2}^{\infty} M_k^{(n)} \lambda^{k-1}\right|$$

$$\leqslant |M_1^{(n)} - \mathrm{Sp}\,A| + \sum_{k=2}^{\infty} |M_k^{(n)} \lambda^{k-1}| \leqslant M_1^{(n)} - \mathrm{Sp}\,A| + \sum_{k=2}^{\infty} |\lambda_1^{(n)}|^{k-1} |\lambda|^{k-1} C_0$$

$$\leqslant |M_1^{(n)} - \mathrm{Sp}\,A| + \frac{|\lambda_1^{(n)}| \cdot |\lambda| C}{1 - |\lambda_1^{(n)}| \cdot |\lambda|}.$$

We now fix λ and let n tend to infinity. Then, as we have shown,

$$|\lambda_1^{(n)}| \to 0.$$

When $n \to \infty$, the number $M_1^{(n)} = \sum_{m=1}^{n} \lambda_m^{(n)} = \operatorname{Sp} AP_n$ tends to $\operatorname{Sp} A$. Indeed for an orthonormal basis $\{\varphi_1, \varphi_2, \dots\}$, with $A = UC$, $C\varphi_k = s_k(A)\varphi_k$, we can write

$$\operatorname{Sp} AP_n = \sum_{k=1}^{\infty} (AP_n\varphi_k, \varphi_k) = \sum_{k=1}^{n} (A\varphi_k, \varphi_k).$$

On the other hand $\operatorname{Sp} A = \sum_{k=1}^{\infty} (A\varphi_k, \varphi_k)$ and consequently $\operatorname{Sp} AP_n \to \operatorname{Sp} A$ as $n \to \infty$. Taking account of all of this, we conclude that

$$\frac{\Delta_n'(\lambda)}{\Delta_n(\lambda)} \to -\operatorname{Sp} A.$$

Suppose that $\operatorname{Sp} A = \alpha \neq 0$. The product $\Delta_n(\lambda)$ converges uniformly because of the estimate proved above:

$$\prod_{j=1}^{n} (1 - \lambda|\lambda_j(A)|) \leqslant \prod_{j=1}^{n} (1 - \lambda s_j(A)), \quad \lambda < 0.$$

Therefore, integrating the limiting relation

$$\lim_{n\to\infty} \frac{\Delta_n'(\lambda)}{\Delta_n(\lambda)} = -\operatorname{Sp} A,$$

we obtain

$$\lim_{n\to\infty} \Delta_n(\lambda) = e^{-\alpha\lambda}.$$

We now estimate the function $\Delta_n(\lambda)$ and its limit, using another representation for $\Delta_n(\lambda)$. Since

$$\Delta_n(\lambda) = \prod_{j=1}^{n} (1 - \lambda\lambda_j^{(n)}),$$

it follows that

$$|\Delta_n(\lambda)| \leqslant \prod_{j=1}^{n} (1 + |\lambda||\lambda_j^{(n)}|) \leqslant \prod_{j=1}^{n} (1 + |\lambda| s_j(AP_n))$$
$$\leqslant \prod_{j=1}^{n} (1 + |\lambda| s_j(A)) \leqslant \prod_{j=1}^{\infty} (1 + |\lambda| s_j(A)).$$

Since the operator A is nuclear, we have $\sum_{j=1}^{\infty} s_j(A) < \infty$. Therefore

$$|\Delta_n(\lambda)| \leqslant \prod_{j=1}^{n}(1+|\lambda|s_j(A)) = \prod_{j=1}^{N}(1+|\lambda|s_j(A)) \prod_{j=N+1}^{\infty}(1+|\lambda|s_j(A)).$$

We now use the elementary inequality

$$1+|\lambda|s_j(A) \leqslant e^{|\lambda|s_j(A)}.$$

Then

$$|\Delta_n(\lambda)| \leqslant \prod_{j=1}^{N}(1+|\lambda|s_j(A))e^{|\lambda|\sum\limits_{j=N+1}^{\infty} s_j(A)}.$$

Since the last estimate is independent of n, it must hold for the function that is the limit of $\Delta_n(\lambda)$. Consequently

$$|e^{-\alpha\lambda}| \leqslant P_N(\lambda)e^{\varepsilon|\lambda|},$$

where $P_N(\lambda) = \prod\limits_{j=1}^{N}(1+|\lambda|s_j(A))$ is a polynomial of degree N in $|\lambda|$ and $\varepsilon = \sum\limits_{j=N+1}^{\infty} s_j(A)$. Choosing N sufficiently large, we can make ε as small as desired. It is obvious that the inequality

$$|e^{-\alpha\lambda}| \leqslant P_N(\lambda)e^{\varepsilon|\lambda|}$$

cannot hold for every $\varepsilon > 0$. Therefore $\alpha = 0$, i.e., $\mathrm{Sp}\, A = 0$ for a Volterra operator A.

We now consider the general case. Let A be an arbitrary completely continuous operator. Let $\bar{L} = \overline{\bigcup\limits_{\lambda_k \neq 0} L_{\lambda_k}}$ be the closed subspace spanned by the set of root vectors of the operator A corresponding to nonzero eigenvalues. Let $\{\varphi_j\}$ be an orthonormal system for which $(A\varphi_j, \varphi_j) = \lambda_j(A)$ (a *Schur system*). Let A_0 be the restriction of the operator A to L. Then

$$\mathrm{Sp}\, A_0 = \sum_j (A\varphi_j, \varphi_j) = \sum_j \lambda_j(A).$$

The subspace $\bar{L}$ is invariant with respect to the operator A. Let P be the projection on $\bar{L}$, i.e. $PH = \bar{L}$ and Q the projection on $\bar{L}^{\perp} = H \ominus \bar{L}$. Then $P + Q = E$, $PQ = QP = 0$, and

$$\begin{aligned} A = (P+Q)(P+Q)A &= (P+Q)A(P+Q) \\ &= PAP + QAP + PAQ + QAQ. \end{aligned}$$

It is obvious that

$$\begin{aligned}\operatorname{Sp} A = \operatorname{Sp} PAP + \operatorname{Sp} QAP + \operatorname{Sp} PAQ + \operatorname{Sp} QAQ \\ = \operatorname{Sp} PAP + \operatorname{Sp} QPA + \operatorname{Sp} APQ + \operatorname{Sp} QAQ,\end{aligned}$$

since $\operatorname{Sp} TH = \operatorname{Sp} HT$ if T is a completely continuous operator and H is bounded. Let us prove this last assertion. Let

$$T = \sum_{j=1}^{\infty} s_j(T)(\cdot, \varphi_j)\psi_j.$$

Then

$$HT = \sum_{j=1}^{\infty} s_j(T)(\cdot, \varphi_j)H\psi_j,$$

$$\operatorname{Sp}(HT) = \sum_{j=1}^{\infty} (HT\varphi_j, \varphi_j) = \sum_{j=1}^{\infty} s_j(T)(H\psi_j, \varphi_j).$$

On the other hand,

$$TH = \sum_{j=1}^{\infty} s_j(T)(\cdot, H^*\varphi_j)\psi_j,$$

$$\operatorname{Sp} TH = \sum_{j=1}^{\infty} (TH\psi_j, \psi_j) = \sum_{j=1}^{\infty} s_j(T)(H\psi_j, \varphi_j),$$

i.e.,

$$\operatorname{Sp}(HT) = \operatorname{Sp}(TH).$$

Taking account of the fact that $QP = 0$, $PQ = 0$, and QAQ is a Volterra operator (cf. Lemma 5 of Sec. 4.3), we obtain

$$\operatorname{Sp} A = \operatorname{Sp} PAP + \operatorname{Sp} QAQ = \operatorname{Sp} PAP.$$

But $\operatorname{Sp} PAP = \operatorname{Sp} A_0 = \sum_j \lambda_j(A)$. Indeed let $H = L + Z$, let $\{\varphi_j\}$ be a Schur basis in L, and let $\{\varphi'_k\}$ be any orthonormal basis of Z. Then

$$\operatorname{Sp} PAP = \sum_j (PAP\varphi_j, \varphi_j) + \sum_k (PAP\varphi'_k, \varphi'_k) = \sum_j (A\varphi_j, \varphi_j),$$

since $P\varphi_k' = 0$ for any k and $(PAP\varphi_j, \varphi_j) = (A\varphi_j, \varphi_j)$ for any j. Finally we have

$$\operatorname{Sp} A = \operatorname{Sp} PAP = \operatorname{Sp} A_0 = \sum_j \lambda_j(A). \blacksquare$$

Thus we have proved that the matrix and spectral trace coincide for both finite-dimensional operators and nuclear operators. The question arises whether there is any analog of these theorems for unbounded operators. In this case the spectral and matrix traces do not exist. For that reason the concept of the so-called "regularized trace" arises. We shall obtain the regularized traces for a large class of operators.

EXAMPLES.

1. A bounded linear operator A is *dissipative* if its imaginary component $A_I = \dfrac{A - A^*}{2i}$ is a nonnegative operator. We remark that $A_R = \dfrac{A + A^*}{2}$ is called the real component. The equality $A = A_R + iA_I$ holds.

Consider the following operator on $L^2(0,1)$.

$$Af = 2i \int_0^x f(t)\, dt.$$

It is obvious that $A^* f = -2i \int_x^1 f(t)\, dt$, $A_I f = \int_0^1 f(t)\, dt$, and $A_R f = i\int_0^x f(t)\, dt - i\int_x^1 f(t)\, dt$. The quadratic form of the operator A satisfies $(Af, f) = \int_0^1 \int_0^1 f(t)\overline{f(s)}\, ds\, dt = \int_0^1 |f(t)|^2\, dt \geqslant 0$. Hence the operator A is dissipative.

2. A linear operator A is called *simple* if it is bounded and has no invariant subspace in common with A^* on which it coincides with A^*. The integral operator of the preceding example is a simple operator.

3. If a dissipative operator A is nuclear, its root vectors form a complete system in the Hilbert space H.

Indeed, according to the trace theorem for nuclear operators

$$\sum_j \lambda_j(A) = \operatorname{Sp} A = \operatorname{Sp} A_R + i \operatorname{Sp} A_I.$$

Consequently, comparing the imaginary parts of this equality, we have

$$\sum_j \operatorname{Im} \lambda_j(A) = \operatorname{Sp} A_I.$$

Let $L_0 = \overline{\bigcup_{\operatorname{Im}\lambda_j \neq 0} L_{\lambda_j}}$ and let A_0 be the restriction of the operator A to L_0. Let $\{\varphi_j\}$ be an orthonormal Schur system such that L_0 is the closed subspace spanned by this system. Suppose $L_0 \neq H$ and so $H = L_0 + L_0^\perp$. Choose an orthonormal basis $\{\varphi'_j\}$ in $L_0^\perp$. Then the systems $\{\varphi_j\}$ and $\{\varphi'_j\}$ together form an orthonormal basis of H. Since the operator A_I is nuclear, we have

$$\operatorname{Sp} A_I = \sum_j (A_I\varphi_j, \varphi_j) + \sum_j (A_I\varphi'_j, \varphi'_j).$$

From this we find that $\sum_j (A\varphi'_j, \varphi'_j) = 0$. The operator A is dissipative and A_I is a nonnegative operator. Therefore each term $(A_I\varphi'_j, \varphi'_j) = 0$. Since in addition the operator the operator A_I is self-adjoint (it is symmetric), there exists a square root of A_I. Then $(A_I\varphi'_j, \varphi'_j) = 0 = (A_I^{1/2} A_I^{1/2} \varphi'_j, \varphi'_j) = (A_I^{1/2}\varphi'_j, A_I^{1/2}\varphi'_j)$, i.e., $A_I^{1/2}\varphi'_j = 0$ and so $A_I\varphi'_j = 0$ for every j. We have now obtained the result that $L_0^\perp \subset Z(A_I)$, where $Z(A_I)$ is the nullspace of the operator A_I. On the subspace $L_0^\perp$, therefore, the operator A satisfies $A = A^* = A_R$, i.e., the operator A is self-adjoint. If $L_0^\perp$ consists of only the zero vector, everything is proved. The system of root vectors corresponding to nonreal eigenvalues is complete. In the opposite case it is necessary to add a basis of $L_0^\perp$ consisting of eigenvectors of the operator A.

3. REGULARIZED SUMS OF ZEROS OF A CLASS OF ENTIRE FUNCTIONS. THE TRACE OF A DIFFERENTIAL OPERATOR

3.1. Functions of Class K

Consider an entire function $f(z)$ that admits a representation of the form

$$f(z) = \sum_{k=0}^{N-1} e^{\alpha_k z} P_{k,h}(z)$$

for each integer $h \geqslant 0$, where α_k are complex numbers and

$$P_{k,h}(z) \sim z^{n_k} \sum_{\nu=0}^{h} \beta_\nu^{(k)} z^{-\nu} + o(z^{n_k - h})$$

as $z \to 0$. In this formula n_k is some integer and $\beta_0^{(k)} \neq 0$.

It is assumed that the z-plane can be covered by a finite number of open sectors containing the origin in each of which the functions $P_{k,h}(z)$ are analytic for $|z| > R$.

In what follows we shall omit the index h of $P_{k,h}(z)$ and write simply

$$P_k(z) \sim z^{n_k} \sum_{\nu=0}^{\infty} \beta_\nu^{(k)} z^{-\nu}, \quad z \to \infty.$$

We shall also assume that the representation for $P_k(z)$ can be differentiated termwise.

We shall agree to call functions with the properties described above *functions of class* K. The numbers α_k and $\beta_\nu^{(k)}$ will be called the *asymptotic parameters* of the function $f(z)$.

Functions of class K arise in solving differential equations containing a parameter z. Consider, for example, a boundary-value problem on the closed interval $[0, 1]$ for the differential equation

$$\frac{d^n y}{dx^n} + a_1(x,z)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n(x,z)y = 0,$$

whose coefficients have the form

$$a_q(x,z) = z^q \sum_{j=0}^{q} z^{-j} a_{q_j}(x), \quad q = 1, 2, \ldots, n.$$

Let the boundary conditions be polynomial functions of z also:

$$U_i(y) = \sum_{\nu=0}^{m_j} z^\nu U_i^\nu(y) = 0, \quad i = 1, 2, \ldots, n,$$

where $U_i^\nu(y)$ are linear forms with respect to the solution $y(x)$:

$$U_i^\nu(y) = \sum_{k=1}^{n} \{a_{ik}^\nu y^{(k-1)}(0) + b_{ik}^\nu y^{(k-1)}(1)\} + \int_0^1 \alpha_i^\nu(x) y(x)\, dx.$$

Let the coefficients of the equation and the functions $\alpha_i^\nu(x)$ be infinitely differentiable with respect to x. If we assume in addition that $a_{q0}(x) = a_{q0}\, r(x)$ $(q = 1, 2, \ldots, n)$, where $r(x) > 0$ and the polynomial $\pi(\lambda) = \lambda^n + a_{10}\lambda^{n-1} + \cdots + a_{n0}$ has no multiple roots, then the equation that determines the eigenvalues of the problem has the form

$$f(z) = 0,$$

where $f(z) \in K$. In doing this it is essential that the asymptotic parameters of $f(z)$ be expressible explicitly in terms of the coefficients of the equation and the coefficients of the boundary conditions.

Our purpose is to obtain explicit expressions for the regularized sums of the zeros of the function $f(z)$ in terms of its asymptotic parameters. That is, we are seeking sums of the form

$$\sum_{(l)} \{z_l^m - A_m(l)\} = s_m \tag{$*$}$$

where z_l are the zeros of the function $f(z)$ and $A_m(l)$ are certain perfectly definite numbers that guarantee the convergence of the series, while m is any natural number.

The formulas $(*)$ can be used to write a system of algebraic equations

$$\sum_{l=1}^{p} z_l^m = s_m^*, \quad m = 1, 2, \ldots, p,$$

connecting the first zeros of $f(z)$. This circumstance is especially crucial in finding the first eigenvalues of boundary-value problems.

The theorems presented below are not connected with differential operators and are of a function-theoretic character. However they make possible a unified method of obtaining the values of regularized sums of eigenvalues of general boundary-value problems for ordinary differential equations of any order.

In this connection we remark that if the coefficients of the equation are only h times differentiable on x, then the function $f(z)$ also admits a representation

$$P_k(z) \sim z^{n_k} \sum_{\nu=0}^{h+n_k} \beta_\nu^{(k)} z^{-\nu}, \quad h + n_k \geqslant 0,$$

as $z \to \infty$. To simplify the exposition all our theorems will be stated in terms of the class K.

3.2. The Zeta-Function

We shall construct a zeta-function associated with $f(z)$. Let $f(z)$ be an entire function of class K. In the complex plane we distinguish the points

$$\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{N-1},$$

and we denote their convex hull by R. In the general case R is an r-gon $(r \leqslant N)$. The directions of the exterior normals to R will be called *critical directions.* Without loss of generality we can assume that the first r exponents

$$\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{r-1}$$

are vertices of the r-gon R.

We remove from the z-plane r closed sectors of arbitrarily small vertex angle having bisectors parallel to the critical directions. The remaining region will be denoted Ω. The region Ω in turn is partitioned into r open sectors Ω_s, $s = 0, 1, \dots, r-1$. The following lemma is easily established.

LEMMA 1. *For sufficiently large M there are no zeros of $f(z)$ in the intersection of Ω and the region $|z| > M$.*

PROOF: In fact it is easy to verify that $\operatorname{Re} \alpha z = (\bar{\alpha}, z)$, where the right-hand side is the inner product of the vectors $\bar{\alpha}$ and z. For definiteness let Z belong to the region Ω_0. Then it is geometrically clear that for $z \notin R$ the inequality $\operatorname{Re} \alpha_0 z - \operatorname{Re} \alpha_k z > \delta|z|$ holds for some $\delta > 0$ ($k \neq 0$). As a consequence we obtain immediately $f(z) = cz^{n_0} e^{\alpha_0 z}(1 + o(1))$. Similar estimates hold in the other sectors Ω_s. ∎

We now choose a ray l in one of the sectors of Ω (for definiteness in Ω_0) and construct a contour Γ_0 consisting of the ray l traversed twice and a circle γ with center at 0. Without loss of generality we can assume that $f(0) \neq 0$ (otherwise $f(z)$ could be divided by an integer power of z). It is obvious that when this is done the ray l and the circle γ can be chosen so that all the zeros of $f(z)$ end up outside the contour Γ_0.

Noting further that for $z \in \Gamma_0$ and $z \to \infty$

$$\frac{f'(z)}{f(z)} = \alpha_0 + \frac{P_0'(z)}{P_0(z)} + O(e^{-\delta|z|}) \sim \sum_{\nu=0}^{\infty} \frac{\omega_\nu^{(0)}}{z^\nu},$$

we consider the integral

$$Z_0(\sigma) = \frac{1}{2\pi i} \int_{\Gamma_0} z^{-\sigma} \frac{f'(z)}{f(z)}\, dz,$$

which converges in the half-plane $\operatorname{Re} \sigma > 1$. In the formula for $Z_0(\sigma)$ we have set

$$z^{-\sigma} = e^{-\sigma \operatorname{Ln} z},$$

where $\operatorname{Ln} z$ is a fixed regular branch of the logarithm on the exterior of Γ_0. We call the function $Z_0(\sigma)$ the *zeta-function* associated with the function $f(z)$.

LEMMA 2. *For $\operatorname{Re} \sigma > 1$*

$$Z_0(\sigma) = \sum_{(l)} z_l^{-\sigma},$$

where z_l are the zeros of $f(z)$.

PROOF: Since $f(z)$ is an entire function of order 1, the ratio $\dfrac{f'(z)}{f(z)}$ has an expansion that is uniformly convergent in every finite disk:

$$\frac{f'(z)}{f(z)} = \sum_{(l)} \left\{ \frac{1}{z - z_l} + \frac{1}{z_l} \right\} + a.$$

Using the fact that the zeros of $f(z)$ for $|z_l| > R$ lie in the sectors, it is not difficult to obtain the estimate $|z - z_l| > \delta |z_l|$ $(\delta > 0)$ for all l and $z \in \Gamma_0$.

We break the sum on the right-hand side into two sums:

$$\sum_{(l)} = \sum_{(l')} + \sum_{(l'')},$$

placing in the second sum the terms for which $|z_l| > R$. It is easy to see that

$$\Big| \sum_{(l'')} \Big| \leqslant \sum_{(l'')} \frac{|z|}{|z - z_l| \cdot |z_l|} < \frac{|z|}{\delta} \sum_{(l'')} \frac{1}{|z_l|^2} < \varepsilon |z|$$

for sufficiently large R. After multiplying $f'(z)/f(z)$ by $z^{-\sigma}$ $(\operatorname{Re}\sigma > 2)$, we integrate both sides over the contour Γ_0. The estimate for the sums enables us to interchange the order of summation and integration. Since we also have

$$\frac{1}{2\pi i} \int_{\Gamma_0} z^{-\sigma} \left\{ \frac{1}{z - z_l} + \frac{1}{z_l} \right\} dz = z_l^{-\sigma},$$

it follows that everything is proved for $\operatorname{Re}\sigma > 2$. Remarking that both sides of the equality being proved are defined and regular in the half-plane $\operatorname{Re}\sigma > 0$ we deduce that the equality is valid for $\operatorname{Re}\sigma > 1$. ■

LEMMA 3. *The zeta-function $Z_0(\sigma)$ can be analytically continued to the whole σ-plane as an entire function.*

PROOF: We break the integral for $Z_0(\sigma)$ into four integrals:

$$\begin{aligned} Z_0(\sigma) = \frac{1}{2\pi i} \oint_{\gamma} z^{-\sigma} \frac{f'(z)}{f(z)}\, dz + \frac{1}{2\pi i} \int_{\Gamma_0'} z^{-\sigma} \left(\frac{f'(z)}{f(z)} - \sum_{\nu=0}^{\nu_0} \frac{\omega_\nu^{(0)}}{z^\nu} \right) dz \\ + \frac{1}{2\pi i} \int_{\Gamma_0} z^{-\sigma} \sum_{\nu=0}^{\nu_0} \frac{\omega_\nu^{(0)}}{z^\nu}\, dz - \frac{1}{2\pi i} \oint_{\gamma} z^{-\sigma} \sum_{\nu=0}^{\nu_0} \frac{\omega_\nu^{(0)}}{z^\nu}\, dz \\ = I_1(\sigma) + I_2(\sigma) + I_3(\sigma) + I_4(\sigma). \end{aligned}$$

It is easy to see that $I_1(\sigma)$ and $I_4(\sigma)$ are entire functions of σ; $I_2(\sigma)$ can be analytically continued to the half-plane $\operatorname{Re}\sigma > -\nu_0$. Finally $I_3(\sigma)$ vanishes for $\operatorname{Re}\sigma > 1$ and consequently can be analytically continued to the whole plane as the zero function. Since ν_0 is arbitrary, Lemma 3 is now proved. ∎

The representation just obtained enables us to find the values of $Z_0(\sigma)$ at the integers.

LEMMA 4. *For $m = 2, 3, \ldots$*

$$Z_0(m) = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} \frac{1}{z^m}\, dz.$$

For $m = 0, 1, 2, \ldots$

$$Z_0(-m) = \omega^{(0)}_{m+1},$$

where $\omega^{(0)}_\nu$ are the coefficients in the expansion of $f'(z)/f(z)$.

PROOF: We shall limit ourselves to the proof of the last equalities, which will be of fundamental importance in what follows. We turn to the formula for $Z_0(\sigma)$. For a nonnegative integer σ we have $I_1(\sigma) = 0$, since the integrand happens to be a regular function of z inside γ. Moreover $I_2(\sigma) = 0$ for any integer σ because a single-valued function of z has been integrated along the ray l in two opposite directions. Further taking account of the fact that $I_3(\sigma) \equiv 0$, we reduce the problem to calculating the integral $I_4(\sigma)$. This obviously will lead us to the formula for $Z_0(-m)$.

The equalities for $Z_0(m)$ are established similarly. We emphasize that the values of $Z_0(\sigma)$ at the positive integers are determined by the behavior of $f(z)$ in a neighborhood of zero, while the values at the negative integers are expressed in terms of the asymptotic parameters as $z \to \infty$. ∎

3.3. Regularized Sums of Zeros of a Function of Class K

We shall now study the asymptotics of the zeros of $f(z)$. In the general case one can assert that the following asymptotic formula holds for zeros of $f(z)$ of large absolute value:

$$z_{n,s} = a_s n(1 + o(1)), \quad a_s = \frac{2\pi i}{\alpha_{s+1} - \alpha_s}.$$

Here $s = 1, 2, \ldots, r-1$ is the index of the sector T_s in which the zeros are located; $\bar\alpha_s$ and $\bar\alpha_{s+1}$ are the vertices of the corresponding sides of the polygon R, and for $s = r-1$ the number α_r is understood to be α_0.

The formula for $z_{n,s}$ turns out to be insufficient to obtain the values of the regularized sums. However, under certain assumptions on the exponents it can be sharpened.

We assume first that only the numbers $\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{r-1}$ lie on the boundary of the polygon, so that the remaining $N-r$ exponents consequently lie inside the polygon. In this case the formula for $z_{n,s}$ can be given the following form:

$$z_{n,s} \sim a_s n\left\{1 + b_s \frac{\ln n}{n} + \frac{c_s}{n} + \sum_{k=1}^{\infty} \frac{R_k^{(s)}(\ln n)}{n^{k+1}}\right\},$$

where $R_k^s(\ln n)$ are polynomials of degree k with respect to $\ln n$. All the coefficients are expressed in terms of the asymptotic parameters of $f(z)$. In particular

$$a_s = \frac{2\pi i}{\alpha_{s+1} - \alpha_s}, \quad b_s = \frac{n_s - n_{s+1}}{2\pi i},$$
$$c_s = \frac{1}{2\pi i}\{(n_s - n_{s+1})\operatorname{Ln} a_s - \operatorname{Ln}\beta_0^{s+1} + \operatorname{Ln}\beta_0^s + \pi i\}.$$

Here the numbers n_s are the exponents in the leading terms of the formula for $P_k(z)$.

The asymptotic formula for $z_{n,s}$ is established by the method of iterations, and we shall not take the time to derive it.

An analogous formula for the zeros of $f(z)$ can be obtained in the case when a side of the polygon R contains not two but three or more exponents (under the assumption, however, that the side is divided into commensurable segments by the corresponding points).

The corresponding asymptotic expansions are analogous to those we have obtained, except that they contain fractional powers n.*

We now obtain the regularized sums of the zeros of the function $f(z)$. For simplicity we shall assume that the zeros $z_{n,s}$ of the function $f(z)$ admit an asymptotic representation in integer powers, though in the case of fractional powers n in the asymptotic representation all the reasoning below remains valid in principle.

Raising both sides of the asymptotic expression for $z_{n,s}$ to the power $-\sigma$, we obtain

$$z_{n,s}^{-\sigma} \sim a_s^{-\sigma} n^{-\sigma}\left\{1 + b_s \frac{\ln n}{n} + c_s \frac{1}{n} + \sum_{k=1}^{\infty} \frac{R_k^{(s)}(\ln n)}{n^{k+1}}\right\}^{-\sigma}.$$

---

*In the case when the sides of the polygon are divided into incommensurable segments by the exponents the reasoning remains basically the same, but the asymptotic formulas for $z_{n,s}$ have a much more complicated form.

Since the known Taylor series of the function $(1+x)^\alpha$ is valid for complex values of x and α, we can represent the third factor on the right-hand side by an asymptotic series and thereby obtain the formula

$$z_{n,s}^{-\sigma} \sim \sum_{k=0}^{\infty} \frac{Q_k^{(s)}(\sigma, \ln n)}{n^{k+\sigma}},$$

where

$$Q_k^{(s)}(\sigma, \ln n) = \sum_{\nu=0}^{k} d_{k,\nu}^{(s)}(\sigma) \ln^\nu n,$$

and $d_{k,\nu}^{(s)}(\sigma)$ are polynomials in σ. In particular

$$d_{0,0}^{(s)}(\sigma) = 1, \quad d_{1,0}^{(s)} = -\sigma c_s, \quad d_{1,1}^{(s)}(\sigma) = -\sigma b_s.$$

We now fix a certain sufficiently large integer τ. It follows from the formula for $z_{n,s}^{-\sigma}$ that the function

$$\Psi_\tau^{(0)}(\sigma) = \sum_{n=1}^{\infty}{}' \sum_{s=0}^{r-1} \left[z_{n,s}^{-\sigma} - \sum_{k=0}^{\tau} \frac{Q_k^{(s)}(\sigma, \ln n)}{n^{k+\sigma}} \right]$$

admits an analytic continuation to the half-plane

$$\operatorname{Re}\sigma > -\tau,$$

since the general term of the series is $O(\ln^{\tau+1} n/n^{-(\tau+1+\sigma)})$. Our goal is to find the numbers

$$\Psi_\tau^{(0)}(-m) \quad (m < \tau),$$

which we shall call the *regularized m-sums of the zeros of $f(z)$*.

We remark that the first index of the zero $z_{n,s}$ is determined by the value of the integer parameter in the asymptotic formula; a finite number of zeros of $f(z)$ may turn out to be unlabeled under such a system of enumeration, or, on the other hand, there may be an excess of a finite number of integer indices. The prime on the summation sign above means that in the first case the unlabeled zeros are included in the sum, and in the second that the first terms in brackets, which are included to use up the extra labels, are considered to be zero.

Taking this remark into account, we shall find the values of the regularized sums. We consider the function

$$\Psi_\tau^{(0)}(\sigma) = \sum_{n=1}^{\infty}{}' \sum_{s=0}^{r-1} \left(\sum_{k=0}^{\tau} \frac{Q_k^{(s)}(\sigma, \ln n)}{n^{k+\sigma}} \right),$$

which is regular for $\operatorname{Re}\sigma > 1$. We have $\Psi_\tau^{(0)}(\sigma) = Z_0(\sigma) - \Phi_\tau^{(0)}(\sigma)$. Since $Z_0(\sigma)$ is an entire function, the function $\Phi_\tau^{(0)}(\sigma)$ can be analytically continued to the half-plane $\operatorname{Re}\sigma > -\tau$ along with $\Psi_\tau^{(0)}(\sigma)$. We remark that $\Phi_\tau^{(0)}(\sigma)$ is expressible in terms of the Riemann zeta-function and its derivatives. In fact

$$\Phi_\tau^{(0)}(\sigma) = \sum_{k=0}^{\tau}\sum_{\nu=0}^{k}\Bigl(\sum_{s=0}^{r-1} a_s^{-\sigma} d_{k,\nu}^{(s)}(\sigma)\Bigr)\sum_{n=1}^{\infty}\frac{\ln^\nu n}{n^{k+\sigma}}$$
$$= \sum_{k=0}^{\tau}\sum_{\nu=0}^{k} D_{k,\nu}^{(0)}(\sigma)(-1)^\nu \varsigma^{(\nu)}(k+\sigma).$$

Since the values of the Riemann zeta-function and its derivatives at the negative integers are known, it is possible to find the values $\Phi_\tau^{(0)}(-m)$ for $m < \tau$. Taking account of the formula for $Z_0(-m)$, we arrive at the following theorem.

THEOREM 1. *For any integer $m < \tau$ the following equalities hold*

$$\sum_{n=1}^{\infty}{}'\sum_{s=0}^{r-1}\Bigl[z_{n,s}^m - \sum_{k=0}^{\tau}\frac{Q_k^{(s)}(-m,\ln n)}{n^{k-m}}\Bigr] = \omega_{m+1}^{(0)} - \Phi_\tau^{(0)}(-m),$$

where $\omega_{m+1}^{(0)}$ are the coefficients of the expansion of $f'(z)/f(z)$ and the numbers $\Phi_\tau^{(0)}(-m)$ are defined by the formula above.

Both terms on the right-hand side depend on the choice of the contour Γ_0 introduced in the definition of the function $Z_0(\sigma)$, while their difference is independent of Γ_0, since the left-hand side is independent of Γ_0. Using this invariance and also the series of linear relations that arise from setting the coefficients at the poles of the ς-function and its derivatives equal to zero, one can obtain a linear recurrence relation to determine the coefficients of the asymptotic expansion of $z_{n,s}$.

We now obtain a system for the first zeros of $f(z)$. Let q_0 be some natural number. Since the general term of the series for Ψ_τ^0 has the order $O(n^{-\tau-1-\sigma}\ \ln^{\tau+1} n)$, we easily find that for all $m \leqslant \tau < 1$

$$\sum_{n=1}^{q}{}'\sum_{s=0}^{r-1} z_{n,s}^m = \sum_{n=1}^{q}\sum_{s=0}^{r-1}\sum_{k=0}^{\tau}\frac{Q_k^{(s)}(-m,\ln n)}{n^{k-m}} + \omega_{m+1}^{(0)}$$
$$- \Phi_\tau^{(0)}(-m) + O(q^{m-\tau}\ \ln^{\tau+1} q).$$

These relations can be regarded as a system of equations in the first zeros of the function $f(z)$. The defect of this system is the indeterminacy in regard to the number of unknowns on the left-hand sides of the formula. We shall now remove this indeterminacy. We remark that

$$Q_k^{(s)}(0, \ln n) = 0, \quad k \geqslant 1, \quad \text{and} \quad Q_0^{(s)}(0, \ln n) = 1.$$

Therefore if we set $m = 0$ in the formula, we obtain on the left-hand side an integer equal to the excess or deficiency in the number of zeros under the given method of enumeration. This integer will be called the *defect of regularization* and denoted $\varkappa$. When this is done, it follows that

$$\varkappa = \omega_1^{(0)} + \frac{r}{2} - \sum_{s=0}^{r-1} b_s \operatorname{Ln} a_s + \sum_{s=0}^{r-1} c_s,$$

where $\operatorname{Ln} a_s$ is understood to be the value of the regular branch of the logarithm fixed by us. Thus the number of unknowns on the left-hand side of the formula is $p = qr + \varkappa$.

Setting $\tau > p$, we rewrite the system for the zeros of $f(z)$ in the form

$$\sum_{l=1}^{p} z_l^m = s_m^*(q), \quad m = 1, 2, \ldots, p.$$

The symbols $s_m^*(q)$ denote the right-hand sides. It follows from our reasoning that they are defined up to $O(q^{-\tau+m} \ln^{\tau+1} q)$.

EXERCISES

1. Let the differential operator L be generated by the following boundary-value problem: $l(y) = -y'' + p(x)y = \lambda y$, $y(0) = y(\pi) = 0$. Let λ_n be its eigenvalues. Prove that

$$\sum_{n=1}^{\infty} (\lambda_n - n^2 - c_0) = \frac{1}{2} c_0 - \frac{p(0) + p(\pi)}{4},$$

where

$$c_0 = \frac{1}{\pi} \int_0^{\pi} p(x)\, dx.$$

2. Let L be the differential operator defined by

$$l(y) = y^{(4)} + p(x)y = \lambda y, \quad y(0) = y''(0) = y(\pi) = y''(\pi) = 0.$$

Let λ_n be its eigenvalues. Prove that

$$\sum_{n=1}^{\infty}(\lambda_n - n^4) = -\frac{p(0)+p(\pi)}{4},$$

if

$$\int_0^{\pi} p(x)\,dx = 0.$$

3. Consider the boundary-value problem

$$(D^2-\alpha^2)^2 y = i\alpha R\{(p(x)-\lambda)(D^2-\alpha^2)y - p''(x)y\},$$
$$y(0) = y'(0) = y(1) = y'(1) = 0,$$

$D = d/dx$, λ is a spectral parameter, α and R are real constants, and $p(x)$ is a real-valued function. Calculate the first regularized sum.

This problem arises in the theory of hydrodynamic stability and is known as the Orr-Sommerfeld problem.

4. Let $f(z)$ be an entire function of class K. Consider the function $\varsigma_s(\sigma) = \sum_{(n)} z_{n,s}^{-\sigma}$, where $z_{n,s}$ is the series of zeros located in one of the sectors T_s. Prove that

$$\varsigma_s(\sigma) = \frac{Z_{s+1}(\sigma) - Z_s(\sigma)}{e^{2\pi i\sigma} - 1}.$$

Therefore $\varsigma_s(\sigma)$ is a meromorphic function.

5. Let L be the operator defined by $l(y) = -y'' + p(x)y = \lambda y$, $y(0) = y(\pi) = 0$. Let $\lambda_n > 0$. Prove that

$$\sum_{n=1}^{\infty}\left(\sqrt{\lambda_n} - n - \frac{c_1}{n} + \frac{2}{\pi}\sqrt{\lambda_n}\arctan\frac{1}{\sqrt{\lambda_n}} - \frac{2}{\pi}\right)$$
$$= \frac{B_2}{2} - c_1\gamma + \int_1^{\infty}\sqrt{\xi}\Big[R(\xi) - \frac{l_0}{\sqrt{\xi}} - \frac{l_1}{\xi} - \frac{l_2}{\xi\sqrt{\xi}}\Big]\,d\xi,$$

where

$$l_0 = \frac{\pi}{2}, \quad l_1 = -\frac{1}{2}, \quad l_2 = -\frac{1}{4}\int_0^{\pi} p(x)\,dx, \quad c_1 = -\frac{2}{\pi}l_2,$$

where B_2 is the Bernoulli number, γ is Euler's constant, and

$$R(\xi) = \sum_{n=1}^{\infty}\frac{1}{\lambda_n + \xi}.$$

4. THE TRACE OF A DISCRETE OPERATOR

As already mentioned above, Theorem 1 proved in the preceding section enables us to obtain formulas for the regularized traces of a large class of problems generated by ordinary differential expressions on a finite closed interval with the spectral parameter occurring in complicated expressions. The problem of obtaining formulas for the regularized trace of a partial differential operator is of particular interest. In this section we shall discuss a solution of this problem based on the theory of perturbations of discrete operators.

We give the following definition.

DEFINITION 1. An operator T on a separable Hilbert space H is called *discrete* if there exists a complex number λ_0 such that $R_{\lambda_0} = (T - \lambda_0 E)^{-1}$ is a completely continuous operator on H.

Thus an operator T is discrete if for some number λ_0 its resolvent is completely continuous.

According to a property of the spectrum of completely continuous operators (cf. Lemma 3 of Sec. 4.3.3) the spectrum of the operator R_{λ_0} consists of at most a countable set of normal eigenvalues having 0 as the only limit point.

Since $S(R_{\lambda_0})$, the spectrum of the operator R_{λ_0}, is the image of the set $S(T)$, the spectrum of the operator T (including the point at infinity) under the mapping $\lambda \mapsto (\lambda - \lambda_0)^{-1}$, it follows that the spectrum of the operator T consists of isolated points having no finite limit points.

According to the considerations of Sec. 4.3 the projection corresponding to the point $\lambda \in S(T)$ coincides with the projection of the operator R_{λ_0} corresponding to the eigenvalue $(\lambda - \lambda_0)^{-1}$. (This can be shown using a change of variable in the integral representation of the projection.)

Thus the eigenspaces of a discrete operator T are finite-dimensional, i.e., every eigenvalue of the operator T has finite multiplicity. From Hilbert's identity for the resolvent we have for any λ in the resolvent set:

$$R_\lambda = R_{\lambda_0}(E + (\lambda - \lambda_0)R_\lambda).$$

Consequently if R_{λ_0} is completely continuous, the operator R_λ is completely continuous for any λ.

As a preliminary we prove several auxiliary propositions that are also of independent interest.

In a separable Hilbert space H we consider a closed operator T. Let the rectifiable contour Γ bounding the region D of the complex plane have the following properties:

a) all the points of this contour are regular values of the operator T;

b) the portion of the spectrum of the operator T inside D consists of a finite number of normal eigenvalues $\lambda_1, \ldots, \lambda_n$.

Suppose the bounded operator P on H satisfies $\max\limits_{\lambda \in \Gamma} \|PR_\lambda(T)\| = q < 1$, where $R_\lambda(T) = (T - \lambda E)^{-1}$ is the resolvent of the operator T. We shall show that in this case all the points of the contour Γ are regular values of the operator $T + P$, and that the following relation holds for the resolvent of the operator $T + P$:

$$R_\lambda(T+P) = R_\lambda(T) + \sum_{k=1}^{\infty} (-1)^k R_\lambda(T)[PR_\lambda(T)]^k,$$

where the operator series on the right-hand side converges like the geometric series $\sum q^k$.

In fact the following equality is obvious.

$$(T + P - \lambda E) = (T + P - \lambda E)(T - \lambda E) R_\lambda(T) = [E + PR_\lambda(T)](T - \lambda E).$$

Since $\|PR_\lambda(T)\| \leqslant q < 1$ for $\lambda \in \Gamma$, the operator $E + PR_\lambda(T)$ is invertible and the relation

$$[E + PR_\lambda(T)]^{-1} = \sum_{k=0}^{\infty} (-1)^k [PR_\lambda(T)]^k$$

holds. Moreover, since the operator $T - \lambda E$ is also invertible for $\lambda \in \Gamma$, using the equality $(AB)^{-1} = B^{-1}A^{-1}$, we obtain the relation for $R_\lambda(T+P)$.

Integrating this equality over the contour Γ after multiplying it by $\dfrac{i}{2\pi}$, we obtain the relation

$$P_\Gamma(T+P) = P_\Gamma(T) + \sum_{k=1}^{\infty} C_k,$$

where

$$P_\Gamma(T+P) = \frac{i}{2\pi} \int_\Gamma R_\lambda(T+P)\, d\lambda,$$

$$P_\Gamma(T) = \frac{i}{2\pi} \int_\Gamma R_\lambda(T)\, d\lambda,$$

$$C_k = \frac{(-1)^k}{2\pi} \int_\Gamma R_\lambda(T)[PR_\lambda(T)]^k \, d\lambda, \quad k = 1, 2, \ldots .$$

Let $C = \sum_{k=1}^{\infty} C_k$.

We note that by the results of Sec. 4.3.3 the operators $P_\Gamma(P+T)$ and $P_\Gamma(T)$ are projections, and by the properties of the contour Γ the operator $P_\Gamma(T)$ is the finite-dimensional projection on the subspace $L_\Gamma = \bigcup_{k=1}^{n} L_{\lambda_k}$, where L_{λ_k} is the root subspace corresponding to the normal eigenvalue $\lambda_k \in D$.

We now establish that the operators C_k and the operator C are finite-dimensional and that each has trace zero.

THEOREM 1. *For any* $k = 1, 2, \ldots$ *the operators* C_k *are finite-dimensional and* $\operatorname{Sp} C_k = 0$.

PROOF: By the Cauchy residue theorem we have

$$C_k = \sum_{j=1}^{n} \operatorname{Res}_{\lambda_j} \{R_\lambda(T)[PR_\lambda(T)]^k\}.$$

In a neighborhood of the point λ_j the operator-valued function $R_\lambda(T)$ has the following Laurent series (cf. Sec. 4.3.3):

$$R_\lambda(T) = \frac{P_{-l}}{(\lambda-\lambda_j)^l} + \cdots + \frac{P_{-1}}{(\lambda-\lambda_j)} + P_0 + P_1(\lambda-\lambda_j) + \cdots,$$

where the operators P_{-m}, $m = 1, 2, \ldots$ are finite-dimensional each having dimension at most $M_j = \dim L_{\lambda_j}$. Using this expansion, we find

$$\operatorname{Res}_{\lambda_j}\{R_\lambda(T)[PR_\lambda(T)]^k\} = \sum_{\substack{i_1+\cdots+i_{k+1}=-1 \\ -l\leqslant i_m}} P_{i_1} P P_{i_2} \cdots P P_{i_{k+1}}.$$

Since the summation extends over the indices i_m satisfying the condition $i_1 + \cdots + i_{k+1} = -1$, each term on the right-hand side of the equality just written contains at least one operator P_m with negative index, i.e., each term is a finite-dimensional operator. Since the number of terms is finite and the number of points of the spectrum of the operator T inside D is also finite, we find that the operators C_k are finite-dimensional. Moreover, using the obvious properties of commutativity and additivity of the trace of finite-dimensional operators and the symmetry with which the indices i_m occur, we have

$$\begin{aligned}\operatorname{Sp} C_k &= \operatorname{Sp} \operatorname{Res}_{\lambda_j}\{R_\lambda(T)[PR_\lambda(T)]^k\} \\ &= \sum_{\substack{i_1+\cdots+i_{k+1}=-1 \\ -l\leqslant i_m}} \operatorname{Sp} P_{i_2} P \cdots P_{i_k} P P_{i_{k+1}} P_{i_1} P \\ &\qquad\qquad = \operatorname{Sp} \operatorname{Res}_{\lambda_j}\{[R_\lambda(T)P]^{k-1} R_\lambda^2(T) P\}.\end{aligned}$$

Hence, applying the Cauchy residue theorem, we obtain

$$\operatorname{Sp} C_k = \frac{i}{2\pi} \operatorname{Sp} \int_\Gamma [R_\lambda(T)P]^{k-1} R_\lambda^2(T) P \, d\lambda.$$

Since the contour Γ is closed and $R_\lambda(T)$ is a differentiable operator-valued function on that contour with $\dfrac{d}{d\lambda} R_\lambda = R_\lambda^2$ (cf. Sec. 4.3.1), we finally have

$$\operatorname{Sp} C_k = \frac{i}{2\pi k} \operatorname{Sp} \int_\Gamma d[R_\lambda p]^k = 0. \blacksquare$$

COROLLARY. *The spectrum of the operator $T + P$ inside D consists of a finite number of normal eigenvalues.*

In fact the operator $P_\Gamma(T+P)$ is finite-dimensional since it is completely continuous, being the limit in norm of completely continuous (even finite-dimensional) operators, and is a projection.

The following theorem holds.

THEOREM 2. *The operator C is finite-dimensional, and* $\operatorname{Sp} C = 0$.

PROOF: We remark that the finite dimensionality of the operator C follows from the same property for the operators $P_\Gamma(T+P)$ and $P_\Gamma(T)$. The assertion that the trace of the operator C equals zero does not follow directly from Theorem 1. To prove it we must first prove several lemmas.

LEMMA 1. *The dimensions of the operators C_k do not exceed* const$\cdot k$, *where the constant is independent of k.*

PROOF: For an arbitrary set of integers $(i_1, i_2, \ldots, i_{k+1})$ we introduce the function $\xi(t) = \sum\limits_{m=1}^{t} i_m$, $t = 1, 2, \ldots, k+1$. We set $\xi(0) = 0$. Since the summation in the formula for $\operatorname{Res}_{\lambda_j} \{R_\lambda(T)[PR_\lambda(T)]^k\}$ extends over sets of indices satisfying the condition $\xi(k+1) = -1$, it follows that for any such set there is an index t_0 such that $\xi(t) \geqslant 0$ for $0 \leqslant t \leqslant t_0$, $\xi(t_0 + 1) < 0$, and $\xi(t_0) < -i_{t_0+1}$. Then each term on the right-hand side of this formula can be represented in the form

$$(P_{i_1} P P_{i_2} \cdots P_{i_{t_0}} P) P_{-m} (P P_{i_{t_0+2}} \cdots P P_{i_{k+1}}),$$

where m is one of the numbers $1, 2, \ldots, l$, $\xi(t) \geqslant 0$ for $0 \leqslant t \leqslant t_0$, and

$$\xi(t_0 + 1) < 0, \quad \xi(t_0) < m, \quad i_{t_0+2} + \cdots + i_{k+1} = m - 1 - \xi(t_0).$$

Conversely each operator of this form is a term on the right-hand side of the formula for $\operatorname{Res}_{\lambda_j}\{R_\lambda(T)[PR_\lambda(T)]^k\}$. We then find that

$$\operatorname{Res}_{\lambda_j}\{R_\lambda(T)[PR_\lambda(T)]^k\} = \Big(\sum_{t_0=0}^{k}\sum_{m=1}^{l}\sum_{t_1=0}^{m-1}\sum_{i_1+\cdots+i_{t_0}=t_1} P_{i_1}P\cdots P_{i_{t_0}}P\Big) \times P_{-m}\Big(\sum_{i_{t_0+2}+\cdots+i_{k+1}=m-1-t_1} PP_{i_{t_0+2}}\cdots PP_{i_{k+1}}\Big).$$

We remark that on the right-hand side of this formula there are $(k+1)\dfrac{l(l+1)}{2}$ terms, each of which is a finite-dimensional operator of dimension at most $M_j = \dim L_{\lambda_j}$, from which Lemma 1 follows. ■

LEMMA 2. *Suppose the sequence of finite-dimensional operators D_n converges in norm to a finite-dimensional operator D and that the relation $\|D_n - D\|\dim D_n \to 0$ holds as $n \to \infty$. Then $\operatorname{Sp} D_n \to \operatorname{Sp} D$ as $n \to \infty$.*

PROOF: We have the decomposition $H = R(D - D_n) \oplus Z(D^* - D_n^*)$, where $R(D-D_n)$ is the range of values of the operator $D - D_n$ and $Z(D^* - D_n^*)$ is the nullspace of the adjoint operator to $D - D_n$. Let $\varphi_1, \ldots, \varphi_{\nu_n}$ be an orthonormal basis of the space $R(D - D_n)$. We complete it to an orthonormal basis of all of H with the elements φ_j, $j = \nu_n + 1, \ldots$ in $Z(D^* - D_n^*)$. Then by the trace theorem for a nuclear operator we have

$$\begin{aligned}\operatorname{Sp}(D - D_n) &+ \sum_{j=1}^{\nu_n}((D - D_n)\varphi_j, \varphi_j) + \sum_{j=\nu_n+1}^{\infty}((D - D_n)\varphi_j, \varphi_j)\\ &= \sum_{j=1}^{\nu_n}((D - D_n)\varphi_j, \varphi_j) + \sum_{j=\nu_n+1}^{\infty}(\varphi_j, (D^* - D_n^*)\varphi_j)\\ &= \sum_{j=1}^{\nu_n}((D - D_n)\varphi_j, \varphi_j).\end{aligned}$$

From this we conclude that

$$\begin{aligned}|\operatorname{Sp} D_n - \operatorname{Sp} D| &\leqslant \|D - D_n\|\dim(D - D_n)\\ &\leqslant \|D - D_n\|(\dim D_n + \dim D) \to 0 \quad \text{as} \quad n \to \infty. \blacksquare\end{aligned}$$

REMARK. In the course of proving Lemma 2 we have established the proposition that the trace of a finite-dimensional operator does not exceed the norm of the operator multiplied by its dimension.

Applying Lemmas 1 and 2 and using the fact that the operator series for C converges geometrically, we obtain the proof of Theorem 2. ■

As an immediate corollary of Theorems 1 and 2 just proved we obtain a well-known theorem on the stability of the root multiplicity in a formulation analogous to the statement of Rouché's theorem on the stability of the number of zeros of an analytic function.

Let $\nu_\Gamma(T) = \sum_{j=1}^{n} \dim L_{\lambda_j}$, and let $\nu_\Gamma(T+P)$ be the analogous quantity for the operator $(T+P)$.

COROLLARY (Stability theorem for root multiplicities). *The following equality holds:*

$$\nu_\Gamma(T) = \nu_\Gamma(T+P).$$

Indeed to prove this equality it suffices to take the trace of both sides of the equality

$$P_\Gamma(T+P) = P_\Gamma(T) + \sum_{k=1}^{\infty} C_k$$

and use the relations

$$\nu_\Gamma(T+P) = \operatorname{Sp} P_\Gamma(T+P), \quad \nu_\Gamma(T) = \operatorname{Sp} P_\Gamma(T).$$

These last relations are obtained by calculating the traces of the corresponding operators in bases whose first vectors form orthonormal bases in the spaces onto which the operators $P_\Gamma(T)$ and $P_\Gamma(T+P)$ project. ■

We now turn to the discussion of the main result of this section.

In a separable Hilbert space H consider a discrete self-adjoint operator T. We require in addition that there exist a real number c such that

$$(Tf, f) \geqslant c(f, f)$$

for all $f \in D_T$, the domain of definition of the operator T. The operators whose quadratic forms satisfy this inequality are said to be *bounded from below*. (If the opposite inequality holds, the operator is *bounded from above*).

In particular if

$$(Tf, f) \geqslant 0$$

for all $f \in D_T$, the operator is called *positive*, just as in the case of a bounded operator. Every operator that is bounded from below or above can be expressed in terms of a positive operator S by means of one of the formulas

$$T = S + cE, \quad T = -S + cE.$$

Therefore it suffices to consider only positive operators.

Since the operator T is discrete, its eigenvalues have a unique limit point at infinity. Since $(Tf, f) \geqslant 0$, it follows that the inequalities

$$0 \leqslant (T\varphi_n, \varphi_n) = \lambda_n(\varphi_n, \varphi_n)$$

hold for the eigenvectors φ_n, i.e., the eigenvalues of a positive discrete operator are nonnegative and can accumulate only at $+\infty$. We arrange them in increasing order, taking account of their possible (finite) multiplicity:

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$$

We denote by $N(\lambda)$ the following function: $N(\lambda) = \sum\limits_{\lambda_n \leqslant \lambda} 1$. Assume that $N(\lambda) = O(\lambda^p)$, $0 < p < 1$ as $\lambda \to \infty$. Let γ be some fixed number satisfying the condition $\gamma > \dfrac{1}{1-p}$. The following lemma holds.

LEMMA 3. *There exists a sequence of real numbers* $\{a_n\}$, $a_n \to \infty$, $n^\gamma \leqslant a_n \leqslant (n+1)^\gamma$ *such that* $d_n = d(a_n, S(T)) \geqslant \mathrm{const} \cdot n^{\gamma(1-p)-1}$, *where the constant is independent of* n *and* $d(\lambda, S(T))$ *denotes the distance from the point* λ *to the spectrum of the operator* T.

PROOF: Consider the closed interval $\Delta_n = [n^\gamma, (n+1)^\gamma]$, $n = 1, 2, \ldots$. The length of the closed interval Δ_n is of order $n^{\gamma-1}$ as $n \to \infty$. It follows from the assumption on $N(\lambda)$ that as $n \to \infty$ the number of points of the spectrum of the operator T lying on the closed interval Δ_n is a quantity of order at most $n^{\gamma p}$. It follows from this that for large indices n there is a point a_n on the closed interval Δ_n lying at distance of order at least $n^{\gamma-1-p\gamma}$ from the spectrum of the operator. The lemma is now proved. ■

We use the notation $l_k = \{\lambda : \operatorname{Re}\lambda = a_k\}$ and let λ_{n_k} and $\lambda_{n_{k+1}}$ be the eigenvalues of the operator T nearest to the line l_k and located on the left and right respectively. According to Lemma 3

$$|\lambda_{n_{k+1}} - \lambda_{n_k}| \geqslant \mathrm{const} \cdot k^{\gamma(1-p)-1} .$$

Let P be some bounded operator defined on all of H. We denote by Γ_k the rectangular contour in the λ-plane with vertices at the points $(-a_k, a_k)$, (a_k, a_k), $(a_k, -a_k)$, $(-a_k, -a_k)$. We prove a lemma.

LEMMA 4. *The operator $T+P$ is a discrete operator. For sufficiently large k all the points of the contour Γ_k are regular points for the operator $T+P$. All the eigenvalues of the operator $T+P$ lying in the strip $-a_k \leqslant \operatorname{Re}\lambda \leqslant a_k$ are inside the rectangle Γ_k, and the operators T and $T+P$ have the same number of eigenvalues (counting multiplicities) inside the contour Γ_k. On the contour Γ_k the following expansion holds:*

$$R_\lambda(T+P) = R_\lambda(T) + \sum_{k=1}^{N} (-1)^k R_\lambda(T)[PR_\lambda(T)]^k + B_N,$$

where

$$B_N = R_\lambda(T+P)[PR_\lambda(T)]^{N+1}.$$

PROOF: We must show that for sufficiently large k the equation

$$(T+P-\lambda E)f = g, \quad \lambda \in \Gamma_k,$$

has a unique solution for any $g \in H$.

Since T is self-adjoint, we have (cf. Sec. 4.3.1) $\|R_\lambda(T)\| = \dfrac{1}{d(\lambda, S(T))}$, where $d(\lambda, S(T))$ is the distance from the point λ to the spectrum $S(T)$ of the operator T. By the choice of the contour Γ_k we have

$$\max_{\lambda\in\Gamma_k} \|R_\lambda(T)\| = O\Big(\frac{1}{d_k}\Big) \quad \text{as } k \to \infty.$$

Since $(1-p)\gamma > 1$, we have $O\Big(\dfrac{1}{d_k}\Big) = o(1)$ as $k \to \infty$. Applying the operator $R_\lambda(T)$ on the right to both sides of the equality $(T+P-\lambda E)f = g$, we obtain $(E + PR_\lambda(T))f = R_\lambda(T)g$. Since P is a bounded operator, it follows that for sufficiently large k we have $\|PR_\lambda(T)\| < 1$ for $\lambda \in \Gamma_k$. It follows from this that the operator $E + PR_\lambda(T)$ is invertible, and on the contour Γ_k the following relation holds for the resolvents,

$$R_\lambda(T+P) = R_\lambda(T) + \sum_{k=1}^{\infty} (-1)^k R_\lambda(T)[PR_\lambda(T)]^k.$$

This relation is important for what we are about to do. One consequence of it is the equality $R_\lambda(T+P) = R_\lambda(T)B_\lambda$, where B_λ is a bounded operator. Since T is a discrete operator, $R_\lambda(T)$ is completely continuous for $\lambda \in \Gamma_k$

and hence $R_\lambda(T+P)$ is also completely continuous for $\lambda \in \Gamma_k$, i.e. the operator $T+P$ is discrete.

The assertion that all the eigenvalues of the operator $T+P$ lying in the strip $-a_k \leqslant \operatorname{Re}\lambda \leqslant a_k$ are inside the rectangle Γ_k for sufficiently large k, as well as the assertion that the operators $T+P$ and T have the same number of eigenvalues inside the contour Γ_k, follows easily from the stability theorem for root multiplicities proved above.

And finally, we verify directly that the expansion for $R_\lambda(P+T)$ is valid by substituting the right-hand side of the equality

$$R_\lambda(T+P) = R_\lambda(T) + \sum_{k=1}^{\infty}(-1)^k R_\lambda(T)[PR_\lambda(T)]^k$$

for $R_\lambda(T+P)$ in the formula for B_N.

The lemma is now proved. ■

We now multiply the relation

$$R_\lambda(T+P) = R_\lambda(T) + \sum_{k=1}^{N}(-1)^k R_\lambda(T)[PR_\lambda(T)]^k + B_N$$

by $\dfrac{\lambda^{m_i}}{2\pi}$ and integrate over the contour Γ_k (here m is a natural number). We have

$$P_{\Gamma_k}(m, T+P) = P_{\Gamma_k}(m,T) + \sum_{\nu=1}^{N}(-1)^\nu C^\nu_{\Gamma_k}(m) + D^{(N)}_{\Gamma_k}(m),$$

where

$$P_{\Gamma_k}(m, T+P) = \frac{i}{2\pi}\int_{\Gamma_k} \lambda^m R_\lambda(T+P)\,d\lambda,$$

$$P_{\Gamma_k}(m, T) = \frac{i}{2\pi}\int_{\Gamma_k} \lambda^m R_\lambda(T)\,d\lambda,$$

$$C^\nu_{\Gamma_k}(m) = \frac{i}{2\pi}\int_{\Gamma_k} \lambda^m R_\lambda(T)[PR_\lambda(T)]^\nu\,d\lambda, \quad \nu = 1,2,\ldots,N,$$

$$D^{(N)}_{\Gamma_k}(m) = \frac{i}{2\pi}\int_{\Gamma_k} \lambda^m R_\lambda(T+P)[PR_\lambda(T)]^{N+1}\,d\lambda.$$

Reasoning as in the proof of Lemma 1 at the beginning of this section we obtain the proof of the following lemma.

LEMMA 5. *The operators* $P_{\Gamma_k}(m, T+P)$, $P_{\Gamma_k}(m, T)$, $C_{\Gamma_k}^{(\nu)}(m)$, *(where* $\nu = 1, 2, \ldots, N$*), and* $D_{\Gamma_k}^{(N)}(m)$ *are finite-dimensional and*

$$\dim D_{\Gamma_k}^{(N)}(m) = O(k^{4\gamma p}) \quad \text{as } k \to \infty$$

(here the symbol dim *denotes the dimension of the operator).*

The following lemma also holds.

LEMMA 6. *The norm of the operator* $D_{\Gamma_k}^{(N)}(m)$ *is of order*

$$O\left(k^{-(N+2)[\gamma(1-p)-1)]+(m+1)\gamma}\right) \quad \text{as } k \to \infty.$$

PROOF: Indeed we have the estimate

$$\|D_{\Gamma_k}^{(N)}(m)\| = \left\| \frac{i}{2\pi} \int_{\Gamma_k} \lambda^m R_\lambda(T+P)[PR_\lambda(T)]^{N+1}\, d\lambda \right\|$$

$$\leqslant \text{const} \cdot \max_{\lambda \in \Gamma_k}(|\lambda^m| \cdot \|R_\lambda(T+P)\| \cdot \|R_\lambda(T)\|^{N+1}) \cdot l(\Gamma_k).$$

We remark that the relation $R_\lambda(T+P) = R_\lambda(T) + \sum_{k=1}^{\infty}(-1)^k R_\lambda(T)[PR_\lambda(T)]^k$ implies that as $k \to \infty$ the norm of the operator $R_\lambda(T+P)$ has the same order of decrease on the contour Γ_k as the norm of the operator $R_\lambda(T)$, which in turn is bounded by the distance from λ to the spectrum $S(T)$ of the operator T. Using Lemma 3 and the properties of the contour Γ_k, we verify Lemma 6 by direct computation. ■

According to Lemma 4 the operators $T+P$ and T have the same number of eigenvalues inside the contour Γ_k for sufficiently large k. Consequently the eigenvalues μ_i of the operator $T+P$ can be enumerated in increasing order of their real parts using the indices from 1 to n_k.

Finally we prove the following lemma.

LEMMA 7. *The following relations hold:*

$$\operatorname{Sp} P_{\Gamma_k}(m, T+P) = \sum_{k=1}^{n_k} \mu_i^m,$$

$$\operatorname{Sp} P_{\Gamma_k}(m, T) = \sum_{i=1}^{n_k} \lambda_i^m,$$

$$\operatorname{Sp} C_{\Gamma_k}^{(\nu)}(m) = -\frac{m}{\nu} \operatorname{Sp} \frac{i}{2\pi} \int_{\Gamma_k} \lambda^{m-1} [R_\lambda(T)P]^\nu\, d\lambda.$$

PROOF: The first two equalities follow easily from the results of Sec. 4.3.3, since these relations constitute the content of the trace theorem for operators on a finite-dimensional space. To prove the last relation it is necessary to repeat the reasoning carried out in the proof of Theorem 1 and apply the formula for integration by parts. ■

Taking the trace of both sides of the equality

$$R_\lambda(T+P) = R_\lambda(T) + \sum_{k=1}^{N}(-1)^k R_\lambda(T)[PR_\lambda(T)]^k + B_N,$$

and applying Lemma 7 and the Cauchy residue theorem, we obtain

$$\sum_{i=1}^{n_k}\Big(\mu_i^m - \lambda_i^m + m \operatorname{Sp} \operatorname{Res}_{\lambda_i}\{\lambda^{m-1}[R_\lambda(T)P]\} + \cdots$$
$$+ \frac{m}{N} \operatorname{Sp} \operatorname{Res}\{\lambda^{m-1}[R_\lambda(T)P]^N\Big) = \operatorname{Sp} D_{\Gamma_k}^{(N)}(m).$$

Using Lemmas 4 and 5 and the remark after Lemma 2, we conclude that as $k \to \infty$ $\operatorname{Sp} D_{\Gamma_k}^{(N)}(m)$ is of order

$$O\left[\frac{1}{k^{(N+2)[\gamma(1-p)-1]-(m+1)\gamma-4\gamma p}}\right].$$

It follows from this that for $N > \dfrac{\gamma(m+1+4p)}{\gamma(1-p)-1} - 2$ we have

$$\operatorname{Sp} D_{\Gamma_k}^{(N)}(m) \to 0 \quad \text{as } k \to \infty.$$

Thus we arrive at the following trace theorem for discrete self-adjoint operators that are bounded above or below.

THEOREM 3. *Let T be a self-adjoint discrete operator that is bounded from below, defined on a separable Hilbert space H, and such that $N(\lambda) = O(\lambda^p)$, $0 < p < 1$. Let γ be some number satisfying the condition $\gamma > 1/(1-p)$, and let P be a bounded operator on H.*

Then the operator $T+P$ is a discrete operator and for the eigenvalues μ_i of the operator $T+P$ and the eigenvalues λ_i of the operator $T+P$ (counting *multiplicities) enumerated in increasing order of their real parts there exists a sequence of natural numbers n_k such that*

$$\lim_{k\to\infty} \sum_{i=1}^{n_k} \Big\{ \mu_i^m - \lambda_i^m + m \operatorname{Sp} \operatorname{Res}_{\lambda_i} [\lambda^{m-1} (R_\lambda(T)P)] + \cdots$$

$$+ \frac{m}{N} \operatorname{Sp} \operatorname{Res}_{\lambda_i} [\lambda^{m-1} (R_\lambda(T)P)^N] \Big\} = 0, \quad \text{for } N > \frac{\gamma(m+1+4p)}{\gamma(1-p)-1} - 2.$$

REMARK. Theorem 3 remains valid in a slightly more general statement. Thus, for example, one can dispense with the hypothesis that the operator T be bounded above or below. The hypothesis that the operator P be bounded can be replaced by the weaker hypothesis that it be "subordinate" to the operator T.

We remark also that many classes of operators defined by boundary-value problems for partial differential equations satisfy the hypotheses of Theorem 3.