

Chapter 6

Distributions. The Fourier Transform

1. DISTRIBUTIONS

1.1. The Concept of a Distribution

In the applied disciplines the term "singular function" is often used. This concept is a generalization of the concept of a function. We now give a typical example of a singular function or, as we shall call it from now on, a distribution.* Let us calculate the density generated by a unit point mass located at the origin. Let a unit of mass be spread uniformly over the interior of a ball of radius ε with center at the origin in \mathbf{R}^3 . Then the average density $f_\varepsilon(x)$ is

$$f_\varepsilon(x) = \begin{cases} \frac{3}{4\pi\varepsilon^3}, & |x| < \varepsilon, \\ 0, & |x| \geq \varepsilon. \end{cases}$$

Let $\delta(x)$ be the unknown density generated by a material point of mass 1. Then obviously for any volume V

$$\int_V \delta(x) dx = \begin{cases} 1, & \text{if } 0 \in V, \\ 0 & \text{if } 0 \notin V. \end{cases}$$

*The Russian name means literally *generalized function* and is sometimes used in English as well, though the term *distribution* used in this translation is more common. *Tr.*

On the other hand if we assume that

$$\delta(x) = \lim_{\epsilon \rightarrow 0} f_\epsilon(x) = \begin{cases} +\infty, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

then, integrating this representation for $\delta(x)$, we arrive at a contradiction, since we have $\int_V \delta(x) dx = 0$. Consequently such a pointwise limit of $f_\epsilon(x)$ cannot be accepted as the definition of the density $\delta(x)$. Let us now regard $f_\epsilon(x)$ as a functional defined by the rule*

$$\int f_\epsilon \varphi dx,$$

where φ is any continuous function. The weak limit of the sequence $f_\epsilon(x)$ as $\epsilon \rightarrow 0$ is obviously

$$\lim_{\epsilon \rightarrow 0} \int f_\epsilon(x) \varphi(x) dx = \varphi(0).$$

Indeed

$$\begin{aligned} \left| \int f_\epsilon(x) \varphi(x) dx - \varphi(0) \right| &= \frac{3}{4\pi\epsilon^3} \left| \int_{|x| < \epsilon} [\varphi(x) - \varphi(0)] dx \right| \\ &\leq \eta \frac{3}{4\pi\epsilon^3} \int_{|x| < \epsilon} dx = \eta. \end{aligned}$$

where $\eta = \max_{|x| < \epsilon} |\varphi(x) - \varphi(0)|$. By the continuity of the function $\varphi(x)$ the number η tends to 0 as ϵ tends to 0.

Thus the weak limit of the function $f_\epsilon(x)$ as $\epsilon \rightarrow 0$ is the functional that assigns to every continuous function $\varphi(x)$ the number $\varphi(0)$:

$$\lim_{\epsilon \rightarrow 0} \int f_\epsilon(x) \varphi(x) dx = (\delta, \varphi) = \varphi(0).$$

The notation (δ, φ) for the functional $\delta(x)$ is convenient and will be used in more general situations. We now calculate the total mass. We have

$$(\delta, \varphi) = (\delta, 1) = 1, \quad \varphi(x) = 1.$$

*From now on we shall omit the limits of integration when the integration extends over the entire space.

The functional $\delta(x)$ is called the *Dirac delta-function*, or simply the *delta-function*.

It was remarked above that to define distributions one must give a definite set of test functions $\{\varphi\}$ in terms of which the distributions are expressed or on which they operate. In doing this it is natural to require that this set of test functions be a vector space with some topology. There exist many spaces of test functions, and the choice of such a space in a particular instance depends on the purpose of the investigation.

1.1.1. The space of test functions K .

This space is made up of the functions φ of compact support having continuous derivatives of all orders. The interval* outside which the function φ vanishes may be different for different functions $\varphi \in K$. The space is a vector space with the usual linear operations. A notion of convergence can also be defined on this space.

A sequence of elements $\{\varphi_n\}$ of K is said to *converge* to the function $\varphi \in K$ if there exists an interval outside which all φ_n vanish and on this interval the sequence of derivatives $\{\varphi_n^{(k)}\}$ converges uniformly to $\varphi^{(k)}$ for each fixed k . If, for example,

$$\varphi(x, a) = \begin{cases} e^{-\frac{a^2}{a^2 - |x|^2}}, & |x| < |a|, \\ 0, & |x| \geq |a|, \end{cases}$$

then $\varphi_n(x) = \frac{1}{n}\varphi(x, a)$ for $n = 1, 2, \dots$ converges to 0 in K as $n \rightarrow \infty$. The functions $\Psi_n(x) = \frac{1}{n}\varphi\left(\frac{x}{n}, a\right)$ do not tend to zero in K .

The convergence in K is generated by a topology on this space that is determined by a system of neighborhoods of zero, each of which is given by a finite set of positive continuous functions $\psi_0(x), \dots, \psi_m(x)$ and consists of the functions of K that satisfy the inequalities

$$|\varphi(x)| < \psi_0(x), \dots, |\varphi^{(m)}(x)| < \psi_m(x)$$

for all x . It is not difficult to verify that this topology generates the convergence in K defined above. We remark that there exist other topologies in K that generate this convergence.

*In the case when the function $\varphi(x)$ is a function of several variables, i.e., $x = \{x_1, x_2, \dots, x_n\}$ belongs to the space \mathbf{R}^n , the interval must be replaced by a bounded region in \mathbf{R}^n .

The following proposition provides many examples of test functions.

PROPOSITION 1. *For a given region G and any $\varepsilon > 0$ there exists an infinitely differentiable function $\eta(x)$ such that $0 \leq \eta(x) \leq 1$, $\eta(x) = 1$ for $x \in G_\varepsilon$, and $\eta(x) = 0$ for $x \notin G_{3\varepsilon}$. The region $G_\varepsilon \supset G$ is the ε -neighborhood of G .*

PROOF: Consider the function

$$f_\varepsilon = \begin{cases} C_\varepsilon e^{-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}}, & |x| \leq \varepsilon, \\ 0, & |x| > \varepsilon. \end{cases}$$

The constant C_ε is determined by requiring $\int f_\varepsilon(x) dx = 1$. If $\chi(x)$ is the characteristic function of the set $G_{2\varepsilon}$, then the function

$$\eta(x) = \int \chi(y) f_\varepsilon(x - y) dy$$

satisfies the required conditions. Indeed the function $\eta(x) = \int_{G_{2\varepsilon}} f_\varepsilon(x - y) dy$ is infinitely differentiable. Further,

$$\begin{aligned} 0 \leq \eta(x) &\leq \int f_\varepsilon(x - y) dy = \int f_\varepsilon(\xi) d\xi = 1; \\ \eta(x) &= \int \chi(y) f_\varepsilon(x - y) dy = \begin{cases} \int f_\varepsilon(\xi) d\xi, & \xi \in G_\varepsilon, \\ 0, & x \notin G_{3\varepsilon}, \end{cases} \end{aligned}$$

which was required. ■

It follows immediately from this proposition that if the region G is bounded, there exists a test function $\eta(x) \in K$ such that $\eta(x) = 1$ for $x \in G_\varepsilon$.

1.1.2. The space of test functions S_∞ .

The space S_∞ consists of the infinitely differentiable functions $\varphi(x)$ on the line that, together with all their derivatives, decrease faster than any power of $|x|^{-1}$, i.e., $\varphi(x) \in S_\infty$ if for any fixed $p, q = 0, 1, \dots$ there exists a constant $C_{p,q}(\varphi)$ such that

$$|x^p \varphi^{(q)}(x)| < C_{p,q}, \quad -\infty < x < \infty.$$

The sequence $\{\varphi_n\}$ is said to converge in S_∞ to $\varphi(x)$ if for any $q = 0, 1, \dots$ the sequence $\{\varphi_n^{(q)}(x)\}$ converges uniformly on any finite interval to $\varphi^{(q)}(x)$ and if the constants $C_{p,q}$ in the inequalities

$$|x^p \varphi_n^{(q)}(x)| < C_{p,q}$$

can be chosen independently of n .

One can introduce the structure of a countably normed space on the space S_∞ by setting

$$\|\varphi\|_n = \sum_{p+q=n} \sup_{\substack{x \in (-\infty, \infty) \\ 0 \leq l \leq p \\ 0 \leq k \leq q}} |(1 + |x|^l) \varphi^{(k)}(x)|.$$

It can be verified that convergence in this countably normed space is equivalent to the definition of convergence in S_∞ introduced above.

We now give a precise definition of a distribution.

DEFINITION 1. A *distribution* is any continuous linear functional on the space of test functions.

The value of a distribution $f(x)$ on a test function φ is denoted (f, φ) . Distributions are also written in the form $f(x)$, where x is the argument of the test functions $\{\varphi(x)\}$.

Thus if a distribution $f(x)$ is defined, then a number (f, φ) is assigned to every test function, and:

1) the distribution is a linear functional, i.e., for any numbers α_1 and α_2 and any test functions $\varphi_1(x)$ and $\varphi_2(x)$

$$(f, \alpha_1 \varphi_1 + \alpha_2 \varphi_2) = \alpha_1 (f, \varphi_1) + \alpha_2 (f, \varphi_2);$$

2) the distribution f is a continuous functional on the space of test functions, i.e., if $\varphi_n \rightarrow 0$ in the space of test functions, then $(f, \varphi_n) \rightarrow 0$ as $n \rightarrow \infty$.

For definiteness we shall take the space K to be the test functions in what follows. The space of distributions over K will be denoted K' .

The distributions given by the formula

$$(f, \varphi) = \int f \varphi dx,$$

where $f(x)$ is a locally integrable function, are called *regular* distributions; all others are called *singular*. If $f(x)$ is a locally integrable function, then

$f(x)$ is also a distribution, since conditions 1) and 2) of the definition of a distribution hold for the functional

$$(f, \varphi) = \int f \varphi dx.$$

In particular passage to the limit is permissible since the integral is taken over a bounded region.

The functional $(\delta, \varphi) = \varphi(0)$ is an example of a singular distribution. Such a functional cannot be represented in the form

$$\int f(x) \varphi(x) dx$$

for any locally integrable function $f(x)$. Indeed, if such a representation were possible, taking $\varphi(x) = e^{-\frac{a^2}{a^2 - |x|^2}}$, we would find that

$$\int_{|x| < |a|} f(x) e^{-\frac{a^2}{a^2 - |x|^2}} dx = \varphi(0) = e^{-1}.$$

But the integral on the left tends to 0 as $a \rightarrow 0$, which is a contradiction.

The space of distributions K' is a vector space. Let $\lambda, \mu \in P$, where P is the scalar field. Then the operation of addition for f and g belonging to K' is defined as follows:

$$(\lambda f + \mu g, \varphi) = \lambda(f, \varphi) + \mu(g, \varphi), \quad \varphi \in K.$$

The functional $\lambda f + \mu g$ is linear. Indeed

$$\begin{aligned} (\lambda f + \mu g, \alpha_1 \varphi_1 + \alpha_2 \varphi_2) &= \lambda(f, \alpha_1 \varphi_1 + \alpha_2 \varphi_2) + \mu(g, \alpha_1 \varphi_1 + \alpha_2 \varphi_2) \\ &= \alpha_1 [\lambda(f, \varphi_1) + \mu(g, \varphi_1)] + \alpha_2 [\lambda(f, \varphi_2) + \mu(g, \varphi_2)] \\ &= \alpha_1 (\lambda f + \mu g, \varphi_1) + \alpha_2 (\lambda f + \mu g, \varphi_2). \end{aligned}$$

Continuity is verified similarly. If $\varphi_n \rightarrow 0$ in K as $n \rightarrow \infty$ then

$$(\lambda f + \mu g, \varphi_n) = \lambda(f, \varphi_n) + \mu(g, \varphi_n) \rightarrow 0.$$

Convergence in the space K' is defined as weak-star convergence, i.e., convergence at each test function. A sequence of distributions f_1, f_2, \dots of K' converges to the distribution $f \in K'$ if for any function $\varphi \in K$

$$(f_n, \varphi) \rightarrow (f, \varphi), \quad n \rightarrow \infty.$$

The introduction of this kind of convergence of the sequence $\{f_n\}$ to the distribution f ($f_n \rightarrow f$) is natural, since the space K' is the dual space to K : the space of continuous linear functionals in which weak convergence is defined.

We remark that it can be shown using the axiom of choice that there exist discontinuous linear functionals on K .

It can be shown that the space K' is a complete space. To be specific the following proposition holds.

PROPOSITION 2. *Suppose the sequence f_1, f_2, \dots of distributions converges in K' . Then the limit function f*

$$(f, \varphi) = \lim_{n \rightarrow \infty} (f_n, \varphi), \quad \varphi \in K,$$

belongs to K' .

If the locally integrable functions $f_n(x)$, $n = 1, 2, \dots$, converge uniformly in every bounded region to a locally integrable function $f(x)$, then the distributions f_n corresponding to them converge to the regular distribution f . In fact

$$(f_n, \varphi) = \int f_n(x) \varphi(x) dx.$$

Since the region of integration is bounded and the convergence $f_n(x) \rightarrow f(x)$ is uniform, one can pass to the limit in the Lebesgue integral. Therefore

$$\lim_{n \rightarrow \infty} (f_n, \varphi) = \lim_{n \rightarrow \infty} \int f_n(x) \varphi(x) dx = \int f(x) \varphi(x) dx = (f, \varphi).$$

Applying other theorems on passage to the limit in Lebesgue integration, one can state other conditions which, when imposed on the locally integrable functions $f_n(x)$, guarantee that the corresponding distributions f_n converge to the limiting function.

1.2. Basic Properties of Distributions

1.2.1. Distributions remain invariant under a linear change of variable. To be specific, if $f \in K'$ and A is a nonsingular linear transformation ($\det \|A\| \neq 0$) then we define the distribution $f(Ay + b)$ for any vector b by the equation

$$(f(Ay + b), \varphi) = \left(f, \frac{\varphi[A^{-1}(x - b)]}{\det \|A\|} \right), \quad \varphi \in K.$$

We remark that if f is locally integrable, we obviously have

$$\begin{aligned}(f(Ay + b), \varphi) &= \int f(Ay + b)\varphi(y) dy = \frac{1}{\det \|A\|} \int f(x)\varphi[A^{-1}(x - b)] dx \\ &= \left(f, \frac{\varphi[A^{-1}(x - b)]}{\det \|A\|}\right),\end{aligned}$$

i.e., we have the equality that we have taken as the definition.

1.2.2. The definition of the product of a distribution and an infinitely differentiable function $a(x)$ is simple, namely:

$$(af, \varphi) = (f, a\varphi), \quad \varphi \in K, \quad a\varphi \in K.$$

If f is a locally integrable function, then

$$(af, \varphi) = \int a(x)f(x)\varphi(x) dx = (f, a\varphi),$$

i.e., the same equality holds, which confirms that the definition just given is appropriate. It is easy to verify that the operation of multiplication by an infinitely differentiable function $a(x)$ is linear and continuous as a mapping of K' into K' :

$$\begin{aligned}a(\lambda f + \mu g) &= \lambda(af) + \mu(ag), \quad f, g \in K', \\ af_n &\rightarrow 0 \text{ in } K \text{ as } n \rightarrow \infty, \text{ if } f_n \rightarrow 0 \text{ in } K' \text{ as } n \rightarrow \infty.\end{aligned}$$

It is difficult to define the product of any two distributions. Even the product of two locally-integrable functions is not necessarily a locally-integrable function. A similar situation arises for distributions. The product of two distributions is not necessarily a distribution. Nevertheless if the functions are chosen in a special way, so that the "irregularity" of one of them is compensated by the "regularity" of the other, then their product is a distribution.

1.2.3. For distributions, as for classical functions, the concept of a derivative is important. For simplicity we consider only the case when $x \in \mathbf{R}^1$.

DEFINITION 2. The *derivative* of the distribution $f \in K'$ is the distribution $g \in K'$ defined by the rule

$$(g, \varphi) = (f, -\varphi'), \quad \varphi \in K.$$

We remark that φ' belongs to K when φ does and the notation makes sense. In what follows the functional g will be denoted

$$f' : g = f' = df/dx.$$

If f is a function having a classical continuous derivative, then

$$(f', \varphi) = \int f' \varphi dx = - \int f \varphi' dx = (f, -\varphi'),$$

so that the notation can be carried over to distributions, and we write

$$(f', \varphi) = - \int f \varphi' dx.$$

It is easily shown that f' is a continuous linear functional of K' . Indeed, it is defined on all of K and is linear, since $-\varphi'$ is a test function. Furthermore if $\varphi_n(x) \rightarrow 0$ in K as $n \rightarrow \infty$, then $-\varphi'_n(x) \rightarrow 0$ in K as $n \rightarrow \infty$. Hence by the continuity of f it follows that

$$(f', \varphi_n) = (f, -\varphi'_n) \rightarrow 0.$$

Thus we arrive at the conclusion that every distribution has a derivative.

EXAMPLES

1. If $f_n \rightarrow f$, $f_n \in K'$, $f \in K'$, then $f'_n \rightarrow f'$ as $n \rightarrow \infty$. Indeed

$$(f'_n, \varphi) = (f_n, -\varphi') = -(f_n, \varphi') \rightarrow -(f, \varphi') = (f', \varphi), \quad n \rightarrow \infty.$$

It is clear that the relation $f_n^{(k)} \rightarrow f^{(k)}$ as $n \rightarrow \infty$ also holds for any $k = 1, 2, \dots$.

2. Let $f_n(x) = \frac{\sin nx}{n}$, $n = 0, 1, 2, \dots$. Then $f_n \rightarrow 0$ uniformly with respect to x as $n \rightarrow \infty$, but $f'(x) = \cos nx$ does not tend to any limit in the classical sense. However the functionals $f_n \in K'$ have the property that all their derivatives $f_n^{(k)}$, $k = 0, 1, \dots$ tend to 0 in K' . Indeed for $k = 0$ we have the original sequence, which obviously tends to zero in K' . Now let $k = 1$. Then

$$(f'_n, \varphi) = -(f_n, \varphi') = -\frac{1}{n} \int \sin nx \varphi'(x) dx \rightarrow 0, \quad n \rightarrow \infty$$

etc.

3. Let $a(x)$ be an infinitely differentiable function and $f \in K'$. Then

$$(af)' = a'f + af'.$$

In fact

$$\begin{aligned} ((af)', \varphi) &= (af, -\varphi') = -(f, a\varphi') = -(f, (a\varphi)' - a'\varphi) \\ &= -(f, (a\varphi)') + (f, a'\varphi) = (f', a\varphi) + (a'f, \varphi) \\ &= (af', \varphi) + (a'f, \varphi) = (af' + a'f, \varphi). \end{aligned}$$

4. Let

$$\theta(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

We shall calculate the derivative of the functional $\theta(x)$. We have

$$\begin{aligned} (\theta'(x), \varphi(x)) &= (\theta(x), -\varphi'(x)) \\ &= -\int_0^\infty \varphi'(x) dx = \varphi(0), \quad \text{i.e., } \theta'(x) = \delta(x). \end{aligned}$$

Similarly

$$\theta'(x-h) = \delta(x-h).$$

5. Let

$$x_+^\lambda = \begin{cases} 0 & x \leq 0, \\ x^\lambda, & x > 0, -1 < \lambda < 0. \end{cases}$$

We shall find the derivative of the functional x_+^λ . We have

$$\begin{aligned} ((x_+^\lambda)', \varphi) &= -(x_+^\lambda, \varphi') = -\int_0^\infty x^\lambda \varphi'(x) dx = -\lim_{\epsilon \rightarrow 0} \int_\epsilon^\infty x^\lambda \varphi'(x) dx \\ &= -\lim_{\epsilon \rightarrow 0} \left\{ x^\lambda [\varphi(x) + c] \Big|_\epsilon^\infty - \int_\epsilon^\infty \lambda x^{\lambda-1} [\varphi(x) + c] dx \right\} \end{aligned}$$

Let $c = -\varphi(0)$. Then

$$((x_+^\lambda)', \varphi) = \lim_{\epsilon \rightarrow 0} \int_\epsilon^\infty \lambda x^{\lambda-1} [\varphi(x) - \varphi(0)] dx.$$

The function defined by this equality is denoted $\lambda x_+^{\lambda-1}$, i.e., $(x_+^\lambda)' = \lambda x_+^{\lambda-1}$. The functional $\lambda x_+^{\lambda-1}$ is not regular, but for $x \neq 0$ it coincides with a regular functional.

6. Suppose the series

$$\sum_{k=1}^{\infty} b_k(x) = s(x)$$

converges uniformly on every bounded region (here $b(x)$ are locally integrable functions). Then this series, formally differentiated any number of times, converges in K' .

Indeed the sequence $s_n(x)$ of partial sums of this series converges uniformly on every bounded region. It follows from the proof of Proposition 2 that the sequence $\{s_n\}$ converges to some s in K' , which was to be proved.

As a corollary we find that if the coefficients of a trigonometric series

$$\sum_{k=-\infty}^{+\infty} c_k e^{ikz}$$

increase no faster than a power of m , i.e.

$$|c_k| \leq c|k|^m, \quad \text{as } |k| \rightarrow \infty,$$

then the series converges in K' . Indeed the series

$$\frac{c_0 x^{m+2}}{(m+2)!} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{+\infty} \frac{c_k}{(ik)^{m+2}} e^{ikz}$$

converges uniformly in every bounded region of the real line. If we then differentiate $m+2$ times, the series will converge in K' . The derivative of order $m+2$ of this last series coincides with the series

$$\sum_{k=-\infty}^{+\infty} c_k e^{ikz}.$$

7. We now obtain formulas for regularizing divergent integrals of locally integrable functions. Let $f(x)$ be a locally integrable function everywhere except at the point 0, and let it have a nonintegrable singularity at 0 such that the function $f(x)|x^m|$ is locally integrable for some integer m . Then the integral

$$(f, \varphi) = \int f \varphi dx,$$

which is in general divergent, admits a regularization, for example, of the form

$$(f, \varphi) = \int f \left\{ \varphi(x) - \left[\varphi(0) + \frac{D\varphi(0)}{1!}x + \cdots + \frac{D^m\varphi(0)}{m!}x^m \right] \theta(1 - |x|) \right\} dx, \\ \theta(1 - |x|) = 1, \quad |x| < 1 \quad \text{and} \quad \theta(1 - |x|) = 0, \quad |x| \geq 1,$$

where D is the differentiation operator.

8. (Approximate identities). It is easy to verify that the following sequences converge to the Dirac delta-function $\delta(x)$ in K' as $\varepsilon \rightarrow +0$:

$$\frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}, \quad \frac{1}{\pi x} \sin \frac{x}{\varepsilon}, \quad \frac{\varepsilon}{\pi x^2} \sin^2 \frac{x}{\varepsilon}, \quad \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{x^2}{4\varepsilon}}.$$

It is possible to give a general criterion for a sequence $f_\varepsilon(x)$ to converge to $\delta(x)$. For this to happen it is obviously necessary that for $N > 0$ the quantities

$$\left| \int_a^b f_\varepsilon(x) dx \right|$$

be bounded by a constant independent of a, b , and ε provided $|a| \leq N$ and $|b| \leq N$, and also that

$$\lim_{\varepsilon \rightarrow 0} \int_a^b f_\varepsilon(x) dx = \begin{cases} 0, & a < b < 0 \text{ or } 0 < a < b, \\ 1, & a < 0 < b. \end{cases}$$

In this case $F_\varepsilon(x) = \int_{-1}^x f_\varepsilon(t) dt$ tends to the function

$$\theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Therefore

$$f_\varepsilon(x) = F'_\varepsilon(x) \rightarrow \theta'(x) = \delta(x), \quad \varepsilon \rightarrow +0.$$

1.3. Differential Equations in Distributions

For distributions one can formally construct differential equations of the form

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_n(x)y(x) = f(x),$$

where $a_i(x)$ are infinitely differentiable functions, $i = 0, 1, \dots, n$, and $y, f \in K'$. The problem of solving such equations in distributions then arises. For the simplest equation

$$y' = 0$$

the only solution in the classical case is a constant. It turns out that the general solution of this equation in the class K' is also

$$y = C = \text{const.}$$

Indeed, we consider a distribution to be zero if for any test function φ

$$(f, \varphi) = 0.$$

Therefore our differential equation can be written in the form

$$(y', \varphi) = -(y, \varphi') = 0.$$

Thus the distribution y is defined by this last equality on the subset of test functions that coincide with the first derivatives of test functions. Naturally the functional can be extended to the whole space K without losing its linearity or continuity. We need to determine the amount of arbitrariness possible when making such an extension.

The test function $\varphi_0(x)$ can be represented as the derivative of a test function if and only if $\int \varphi_0 dx = 0$. Indeed suppose $\varphi_0(x) = \varphi_1'(x)$. Then by the fact that $\varphi_1(x) \in K$ has compact support we have

$$\int \varphi_0(x) dx = \int \varphi_1'(x) dx = 0.$$

Conversely suppose $\int \varphi_0(x) dx = 0$. Then $\varphi_1(x) = \int_{-\infty}^x \varphi_0(x) dx$ is a test function and $\varphi_1'(x) = \varphi_0(x)$, which was to be proved.

We now study the extension of the functional y to the entire space K . For any test function $\varphi(x)$ we have the equality

$$\varphi(x) = \varphi_1(x) \int \varphi(x) dx + \varphi_0(x),$$

where $\varphi_1(x)$ is any test function such that

$$\int \varphi_1(x) dx = 1.$$

In fact, let $\varphi(x)$ be an arbitrary test function. Then

$$\varphi_0(x) = \varphi(x) - \varphi_1(x) \int_{-\infty}^{+\infty} \varphi(x) dx$$

is also a test function, being the difference of two test functions. Moreover $\int \varphi_0(x) dx = 0$.

Taking account of the equality

$$\varphi(x) = \varphi_0(x) + \varphi_1(x) \int \varphi(x) dx$$

and the fact that the functional y is already defined on $\varphi_0(x)$, we conclude that to extend y it remains only to define it on the function $\varphi_1(x)$. We set $(y, \varphi_1) = C = \text{const.}$ Then

$$(y, \varphi) = (y, \varphi_0) + (y, \varphi_1) \int \varphi(x) dx = \int C \varphi(x) dx.$$

Thus,

$$(y - C, \varphi) = 0, \quad \varphi \in K,$$

i.e.,

$$y = C = \text{const.}$$

The functional y is now defined unambiguously on any test function:

$$(y, \varphi) = (y, \varphi_1) \int \varphi(x) dx.$$

It is obviously linear and continuous. Therefore the equation $y' = 0$ has no solutions except the classical ones, i.e., the constants.

A distribution g that is a solution of the equation

$$g' = f$$

in the class K' is called a *primitive* of the distribution f . Every distribution has a primitive that is unique up to an additive constant. We rewrite the equation for the primitive in the equivalent form

$$(g, -\varphi') = (f, \varphi).$$

As before, we see that the functional g is defined on the subset of the test functions that coincide with the first derivatives of test functions. We extend

the functional g to the whole space K . As above we make use of the equality for any test function φ :

$$\varphi(x) = \varphi_1(x) \int \varphi(x) dx + \varphi_0(x),$$

where $\varphi_1(x)$ is a test function for which $\int \varphi_1(x) dx = 1$ and $\int \varphi_0(x) dx = 0$. Then

$$(g, \varphi) = (g, \varphi_1) \int \varphi(x) dx + (g, \varphi_0).$$

We set $(g, \varphi_1) = 0$. In this way the functional g is defined on any test function φ and

$$(g, \varphi) = (g, \varphi_0) = -\left(f, \int_{-\infty}^x \varphi_0(t) dt\right).$$

Indeed on the functions $\{\hat{\varphi}\}$ which can be represented as the derivatives of other functions the functional g is defined and

$$(g, \hat{\varphi}') = -(f, \hat{\varphi}).$$

Therefore if $\varphi_0 = \hat{\varphi}'$, then $\hat{\varphi} = \int_{-\infty}^x \varphi_0(t) dt$ and

$$(g, \varphi) = -(f, \hat{\varphi}) = -\left(g, \int_{-\infty}^x \varphi_0(t) dt\right).$$

Furthermore for this functional g

$$(g', \varphi) = (g, -\varphi') = -(g, \varphi') = \left(f, \int_{-\infty}^x \varphi'(t) dt\right) = (f, \varphi)$$

since we can assume that $\varphi_0 = \varphi'$. Thus the functional g satisfies the original equation. It is obviously linear and continuous. This primitive, i.e., particular solution of the equation $g' = f$, is determined up to an additive constant—the general solution of the homogeneous equation $g' = 0$. It has thereby been proved that every distribution has a primitive.

The results just obtained can be easily carried over to systems of n linear ordinary differential equations in n unknown functions.

1.4. The Tensor Product and Convolution of Distributions

We shall define two more important operations for distributions—the tensor product and convolution. If $f(x)$ and $g(y)$ are locally integrable functions (say on \mathbf{R}^n and \mathbf{R}^m respectively), the function $f(x)g(y)$ is also locally

integrable (on \mathbf{R}^{n+m}). It determines a regular distribution acting on test functions $\varphi(x, y)$ according to the formulas

$$\begin{aligned}(f(x)g(y), \varphi(x, y)) &= \iint f(x)g(y)\varphi(x, y) dx dy \\ &= \int f(x) \left(\int g(y)\varphi(x, y) dy \right) dx = (f(x), (g(y), \varphi(x, y))), \\ (g(y)f(x), \varphi(x, y)) &= \iint g(y)f(x)\varphi(x, y) dx dy \\ &= \int g(y) \left(\int f(x)\varphi(x, y) dx \right) dy = (g(y), (f(x), \varphi(x, y))).\end{aligned}$$

These equalities constitute the assertion of Fubini's theorem.

The first of these equalities is taken as the definition of the *tensor product* $f(x) \cdot g(y)$ of the distributions $f(x) \in K'(\mathbf{R}^n)$ and $g(x) \in K'(\mathbf{R}^m)$. By definition we set

$$(f(x) \cdot g(y), \varphi(x, y)) = (f(x), (g(y), \varphi(x, y))), \quad \varphi \in K(\mathbf{R}^{n+m}).$$

It can be verified that the right-hand side of this last equality defines a continuous linear functional on $K(\mathbf{R}^{n+m})$.

The tensor product is a commutative operation, i.e.

$$f(x) \cdot g(y) = g(y) \cdot f(x),$$

where $g(y) \cdot f(x)$ is defined analogously. The tensor product is linear and continuous with respect to f as a mapping from $K'(\mathbf{R}^n)$ to $K'(\mathbf{R}^{n+m})$ and with respect to g as a mapping from $K'(\mathbf{R}^m)$ to $K'(\mathbf{R}^{n+m})$. This operation is associative, i.e.

$$f(x) \cdot [g(y) \cdot h(z)] = [f(x) \cdot g(y)] \cdot h(z).$$

The tensor product can be differentiated according to the rule

$$D_x^\alpha [f(x) \cdot g(y)] = D^\alpha f(x) \cdot g(y),$$

and also multiplied by a function $a(x) \in C^\infty(\mathbf{R}^n)$

$$a(x)[f(x) \cdot g(y)] = (a(x)f(x)) \cdot g(y).$$

Translation is also defined for the tensor product

$$(f \cdot g)(x + h, y) = f(x + h) \cdot g(y).$$

We now define the convolution of distributions. Let $f(x)$ and $g(x)$ be locally integrable functions in \mathbf{R}^n such that the function $h(x) = \int |g(y)f(x-y)| dy$ is also locally integrable. Then the convolution $f * g$ of these two functions is defined as the function

$$f * g(x) = \int f(y)g(x-y) dy = \int g(y)f(x-y) dy = (g * f)(x).$$

If either of the convolutions $f * g$ and $|f| * |g| = h$ exists, the other does, and the inequality $|(f * g)(x)| \leq h(x)$ holds for almost all x , so that the convolution $f * g$ is also a locally integrable function on \mathbf{R}^n . Therefore it defines a regular distribution acting on the test functions $\varphi \in K(\mathbf{R}^n)$ according to the rule

$$\begin{aligned} (f * g, \varphi) &= \int (f * g)(t)\varphi(t) dt = \int \left[\int g(y)f(t-y) dy \right] \varphi(t) dt \\ &= \int g(y) \left[\int f(t-y)\varphi(t) dt \right] dy = \int g(y) \left[\int f(x)\varphi(x+y) dx \right] dy. \end{aligned}$$

Here we have used Fubini's theorem. Thus in this case

$$(f * g, \varphi) = \iint f(x)g(y)\varphi(x+y) dx dy, \quad \varphi \in K(\mathbf{R}^n).$$

We note that the function $h(x)$ is locally integrable; therefore the convolution exists and is defined by the formula given above if, for example, the functions f and g are integrable on \mathbf{R}^n or if one of them is of compact support.

We now turn to the definition of the convolution for distributions.

We say that a sequence $\varphi_k(x) \in K(\mathbf{R}^n)$ converges to 1 in \mathbf{R}^n if the following conditions hold. For every closed bounded set F there exists an index N such that $\varphi_k(x) \equiv 1$ for $x \in F$ and $k \geq N$, and the functions $\varphi_k(x)$ are uniformly bounded on \mathbf{R}^n , together with all their derivatives, by a constant independent of k .

The equality defining the convolution can now be written in the form

$$(f * g, \varphi) = \lim_{k \rightarrow \infty} (f(x) \cdot g(y), \varphi_k(x, y)\varphi(x+y)),$$

where $\varphi_k(x, y)$ is any sequence that converges to 1 in \mathbf{R}^{2n} . Now let f and g be two distributions such that the following limit exists for any sequence $\varphi_k(x) \rightarrow 1$ in \mathbf{R}^{2n} and is independent of the choice of the sequence:

$$\begin{aligned} (f * g, \varphi) &= \lim_{k \rightarrow \infty} (f(x) \cdot g(y), \varphi_k(x, y)\varphi(x+y)) \\ &= (f(x) \cdot g(y), \varphi(x+y)), \quad \varphi \in K(\mathbf{R}^n). \end{aligned}$$

Then the functional $f * g$ defined by this relation is called the *convolution* of the distributions f and g . We remark that $\varphi(x + y)$ does not in general belong to $K(\mathbf{R}^{2n})$ since it does not have compact support in \mathbf{R}^{2n} . Therefore the convolution of two generalized functions does not always exist.

It can be shown that $f * g \in K'(\mathbf{R}^n)$ and that $f * \delta = \delta * f = f$ for any $f \in K'$. The convolution $f * g$ is a linear mapping from K' to K' with respect to f and g separately. The operation of convolution is commutative. The following relations hold:

$$\begin{aligned} D^n f * g &= D^n (f * g) = f * D^n g, \quad n = 1, 2, \dots, \\ f(x + h) * g(x) &= (f * g)(x + h), \quad h \in \mathbf{R}^n. \end{aligned}$$

2. THE FOURIER TRANSFORM

2.1. The Fourier Transform of Functions of the Space L^1

Assume at first that the function $f(x)$ is integrable on the real line. Then to this function one can assign the function $F(f) = g(\sigma) = \int f(\xi) e^{-i\sigma\xi} d\xi$, which is a bounded continuous function of σ and satisfies $\lim_{|\sigma| \rightarrow \infty} g(\sigma) = 0$. Indeed the fact that $g(\sigma)$ is bounded follows from the estimate

$$|g(\sigma)| \leq \int_{-\infty}^{+\infty} |f(\xi)| d\xi.$$

The fact that $g(\sigma)$ is continuous and tends to zero as $|\sigma| \rightarrow \infty$ is proved by the following reasoning: let $f(x)$ be the characteristic function of an interval (α, β) ; then

$$F(f) = g(\sigma) = \int_{\alpha}^{\beta} e^{-i\sigma x} dx = \frac{e^{-i\sigma\alpha} - e^{-i\sigma\beta}}{i\sigma}$$

and in this case everything is proved. For linear combinations of characteristic functions the assertion is again obviously true. Finally every function f in L^1 is the limit (in the norm of L^1) of linear combinations of characteristic functions. The inequality proved above shows that the corresponding function $g(\sigma)$ is the uniform limit of continuous functions that tend to zero at infinity. Hence the function $g(\sigma)$ is also a continuous function tending to zero at infinity.

The function $F(f) = g(\sigma)$ is called the *Fourier transform* of the integrable function $f(x)$.

We shall now prove a proposition on the existence of the inverse transformation $F^{-1}(g) = f(x)$ —the so-called “inversion formula” given by the rule

$$F^{-1}(g) = f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\sigma) e^{i\sigma x} d\sigma.$$

Assume that the function satisfies the Dini conditions, i.e., there exists $\delta > 0$ such that

$$\int_{-\delta}^{\delta} \frac{|f(x+t) - f(x)|}{|t|} dt < \infty.$$

Then

$$\begin{aligned} f_N(x) &= \frac{1}{2\pi} \int_{-N}^N g(\sigma) e^{i\sigma x} d\sigma = \frac{1}{2\pi} \int_{-N}^N \left\{ \int_{-\infty}^{+\infty} f(\xi) e^{i\sigma(x-\xi)} d\xi \right\} d\sigma \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi) \left\{ \int_{-N}^N e^{i\sigma(x-\xi)} d\sigma \right\} d\xi = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(\xi) \frac{\sin N(x-\xi)}{x-\xi} d\xi \\ &= \frac{1}{\pi} \int_{-\infty}^{+\infty} f(x+t) \frac{\sin Nt}{t} dt. \end{aligned}$$

In reversing the order of integration above we used the fact that the inner integral converges uniformly on the parameter σ . Furthermore

$$f_N(x) - f(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} [f(x+t) - f(x)] \frac{\sin Nt}{t} dt,$$

since $\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin Nt}{t} dt = 1$. In the relation for $f_N(x) - f(x)$ we break the integral of integration into two parts: $|t| \leq M$ and $|t| > M$. Then the term corresponding to the second path can be written in the form

$$\int_{|t|>M} f(x+t) \frac{\sin Nt}{t} dt - f(x) \int_{|t|>M} \frac{\sin Nt}{t} dt.$$

Both integrals converge, and for fixed x this quantity, for M sufficiently large, becomes as small as desired independently of $N > 1$.

The term corresponding to the first path of integration looks as follows:

$$\int_{-M}^M \frac{f(x+t) - f(x)}{t} \sin Nt dt.$$

By the Dini condition the function $\frac{f(x+t) - f(x)}{t}$ is integrable; breaking $\sin Nt$ into two exponentials, in accordance with what was said above we

conclude that this quantity also tends to zero as $N \rightarrow \infty$ (cf. the behavior of the Fourier transform as $|\sigma| \rightarrow \infty$). Thus if the function $f(x)$ is integrable and satisfies Dini's condition, then first of all the operator $F(f) = g(\sigma)$ is defined and second, the inverse operator $F^{-1}(g) = f$ exists, which was to be proved.

The following properties of the Fourier transform are easily established starting from the definition of the operator F .

2.1.1. Let $P\left(\frac{d}{dx}\right)$ be a polynomial with constant coefficients in the differentiation operator $\frac{d}{dx}$, $P\left(\frac{d}{dx}\right) = \sum_{0 \leq k \leq n} a_k \frac{d^k}{dx^k}$, $a_k \in P$, the scalar field. Then using integration by parts we conclude that if the function $\varphi(x)$ has integrable derivatives up to order n , then

$$F\left(P\left(\frac{d}{dx}\right)\varphi\right) = P(i\sigma)F(\varphi).$$

2.1.2. By Fubini's theorem we conclude that if f_1 and f_2 are two integrable functions and $f_1 * f_2$ their convolution, then

$$F(f_1 * f_2) = g_1(\sigma)g_2(\sigma), \quad g_1(\sigma) = F(f_1), \quad g_2(\sigma) = F(f_2).$$

2.1.3. Let the functions $f(x)$, $xf(x)$, \dots , $x^n f(x)$ be integrable on the entire real line. Then the formula $F(f) = g(\sigma)$ can be differentiated n times on σ , and by the uniform and absolute convergence of the integrals we have the formula

$$i^k F^{(k)}(f) = g^{(k)}(\sigma) = F(x^k f), \quad k = 0, 1, \dots, n.$$

The functions $g^{(k)}(\sigma)$ are continuous and tend to zero as $|\sigma| \rightarrow \infty$. For an arbitrary polynomial $P(x)$ of degree at most n and constant coefficients we have

$$P\left(i\frac{d}{d\sigma}\right)F(\varphi) = F(P(x)\varphi).$$

Thus the stronger the condition of decrease at infinity imposed on the function $f(x)$ the more smoothness the function $g(\sigma)$ possesses.

If we consider the class S_∞ (cf. Sec. 1.1, Example 2) we find that under the Fourier transform it maps onto the entire class S_∞ (in the argument σ). Indeed, on the one hand we easily conclude that the functions $x^k f^{(q)}(x)$ are bounded and integrable on the entire real line for any $k, q = 0, 1, \dots$

$$\left(|x^{k+2} f^{(q)}(x)| < C_{k+2,q}, \quad |x^k f^{(q)}(x)| < \frac{C_{k+2,q}}{x^2}\right).$$

Then the function $g(\sigma)$ is infinitely differentiable and

$$i^q g^{(q)}(\sigma) = F(x^q f(x)).$$

Furthermore the function $x^q f(x)$ is infinitely differentiable together with $f(x)$, and all its derivatives are integrable, since by Leibniz' formula they are expressible in terms of the integrable functions $x^j f^{(q-j)}(x)$. Therefore the functions

$$(i\sigma)^k g^{(q)}(\sigma) = (-i)^q F[(x^q \cdot f(x))^{(k)}],$$

being the Fourier transforms of integrable functions, are bounded for all k and q , i.e., $g(\sigma) \in S_\infty$.

Thus if $f(x) \in S_\infty$ then $g(\sigma) \in S_\infty$, i.e., $F(S_\infty) \subset S_\infty$ (in the argument σ).

On the other hand to each function $g(\sigma) \in S_\infty$ there corresponds a function $f(x) = F^{-1}(g) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\sigma) e^{i\sigma x} d\sigma$. The function $2\pi f(-x)$ is the Fourier transform of the function $g(\sigma)$ and therefore belongs to S_∞ . But then obviously the function $f(x)$ also belongs to S_∞ and the function $g(\sigma)$ is the Fourier transform of the function $f(x)$. Thus

$$F(S_\infty) \subset S_\infty \quad (\text{in the argument } x).$$

We have proved that $F(S_\infty) = S_\infty$; the operator F establishes a one-to-one correspondence between these classes. We shall use this fact to define the Fourier transform for distributions.

We have now given a complete discussion of the case of an integrable function $f(x)$.

2.2. The Fourier Transform of Functions in the Space L^2

Now consider the case of a function $f(x) \in L^2(-\infty, \infty)$. We remark that a function $f \in L^2(-\infty, \infty)$ need not belong to $L^1(-\infty, \infty)$ (for example, $(1+x^2)^{-1/2}$) and therefore the concept of the Fourier transform needs to be extended to this class of functions.

Let $f \in L^2(-\infty, \infty)$ and

$$g_N(\sigma) = \int_{-N}^N f(x) e^{i\sigma x} dx.$$

Then it can be shown that $g_N(\sigma) \in L^2(-\infty, \infty)$ for any N and it turns out that there exists an element $g \in L^2(-\infty, \infty)$ for which

$$\|g - g_N\|_{L^2} \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

The element $g \in L^2(-\infty, \infty)$ is called the *Fourier transform* of the function $f \in L^2(-\infty, \infty)$:

$$g(\sigma) = F(f).$$

The basic fact about the Fourier transform of functions in $L^2(-\infty, \infty)$ is that

$$\|g(\sigma)\|_{L^2}^2 = 2\pi\|f\|_{L^2}^2,$$

i.e.,

$$\int_{-\infty}^{+\infty} |g(\sigma)|^2 d\sigma = 2\pi \int_{-\infty}^{+\infty} |f(x)|^2 dx.$$

This theorem is proved according to the following scheme. If f belongs to the class S_∞ , the assertion is obvious. Since the functions of S_∞ are dense in $L^2(-\infty, \infty)$, the equality extends by continuity to all of $L^2(-\infty, \infty)$. Indeed if f has compact support and belongs to L^2 , then $f \in L^1(-a, a)$, where $(-a, a)$ is an interval containing the support of $f(x)$. Therefore the following integral exists:

$$g(\sigma) = \int_{-\infty}^{+\infty} f(x)e^{-i\sigma x} dx.$$

Let $\{f_n\}$ be a sequence of functions in S_∞ that vanish outside $(-a, a)$ and $\|f_n - f\|_{L^2} \rightarrow 0$ as $n \rightarrow \infty$. The sequence $g_n = F(f_n)$ converges uniformly to g and is fundamental in $L^2(-\infty, \infty)$, since by what has been said

$$\|g_n - g_m\|_{L^2} = \sqrt{2\pi}\|f_n - f_m\|_{L^2}.$$

Since L^2 is complete, $\{g_n\}$ converges in L^2 and to the same limit to which g_n converges uniformly. Therefore we can pass to the limit in the equality

$$\|g_n\|^2 = 2\pi\|f_n\|^2$$

and in the case of a function of compact support $f(x) \in L^2$ everything is proved.

If f is an arbitrary function in $L^2(-\infty, \infty)$, then the function

$$f_N(x) = \begin{cases} f(x), & |x| \leq N, \\ 0, & |x| > N, \end{cases}$$

belongs to $L^1(-\infty, \infty)$, $g_N(\sigma) = F(f_N)$ exists, and by what has been proved

$$\|g_N - g_M\|_{L^2} = 2\pi\|f_N - f_M\|_{L^2}.$$

Consequently the functions g_N also converge in L^2 to some limit g .

Passing to the limit as $N \rightarrow \infty$ in the relation

$$\|g_N\|^2 = 2\pi\|f_N\|^2,$$

we again obtain the required relation.

We remark that if $f \in L^2(-\infty, \infty) \cap L^1(-\infty, \infty)$, then

$$g(\sigma) = F(f)$$

exists in the ordinary sense. The functions f_N converge to f in L^1 and $g_N(\sigma) \rightarrow g(\sigma)$ uniformly. Moreover $g_N \rightarrow g$ in $L^2(-\infty, \infty)$ also, where g is an element of L^2 . It follows from this that $g(\sigma)$ and this L^2 limit g coincide.

2.3. The Fourier Transform of a Distribution

We now turn to the definition of the Fourier transform for distributions. As the space of test functions we take the space S_∞ . Let S'_∞ be the corresponding space of distributions. Then the *Fourier transform* of the distribution $f \in S'_\infty$ is defined as the continuous linear functional $\psi \in S'_\infty$ defined by the formula

$$(\psi, g) = 2\pi(f, \varphi), \quad g = F(\varphi).$$

Thus

$$(F(f), g) = 2\pi(f, \varphi) = 2\pi(f, F^{-1}(g)),$$

i.e., $F(f)$, $f \in S_\infty$ is the functional that assumes at the element $g \in S_\infty$ the value equal to the value of the original functional multiplied by 2π at the element $\varphi = F^{-1}g \in S_\infty$. The elements $g = F(\varphi)$ range over the entire space S_∞ as φ ranges over the space S_∞ , i.e., the functional $F(f)$ is defined on all of S_∞ and it is obviously linear and continuous.

If $f \in S_\infty$ and $\varphi \in S_\infty$, then as we have already shown

$$2\pi(f, \varphi) = (\psi, g),$$

and for a fixed f there exists only one function ψ (up to equivalence almost everywhere) such that this relation holds for all $\varphi \in S_\infty$. Passing to the limit, we verify that this equality holds also for $f \in L^1(-\infty, \infty)$, i.e., the definition of the Fourier transform that we have given for functionals is an extension of this concept for integrable functions.

If we take the space K , for example, as the space of test functions, then four spaces will occur in the definition of the Fourier transform:

$$K, \quad K', \quad F(K), \quad F(K').$$

EXAMPLES

1. If the function $f(x)$ is such that $f(x)e^{b|x|} \in L^1$ for $b > 0$, for example, if $f(x)$ is of compact support, then its Fourier transform $F(f) = g(\sigma)$ is an analytic function in the open strip: $|\tau| < b$, $\sigma = s + i\tau$. Indeed

$$g(\sigma) = g(s + i\tau) = \int_{-\infty}^{+\infty} f(x)e^{-isz} e^{-\tau z} dx = \int_{-\infty}^{+\infty} f(x)e^{-i\sigma z} dx.$$

The integral for $g(\sigma)$ converges for $|\tau| < b$. Formal differentiation gives

$$g'(\sigma) = \int_{-\infty}^{+\infty} f(x)e^{-i\sigma z} (-ix) dx,$$

and the integral obtained converges uniformly in a sufficiently small neighborhood of a point σ lying inside the strip $|\tau| < b$ of the σ -plane. Thus the function $g(\sigma)$ has a derivative at every point of the open strip $|\tau| < b$, i.e., it is analytic in this region.

2. On the space $L^2(-\infty, \infty)$ the Fourier transform is a bounded linear operator $F : L^2 \rightarrow L^2$. By direct computation one can verify that the functions $\varphi_n = \omega_n e^{-x^2/2}$, where

$$\omega_n(x) = x^n - \frac{n(n-1)}{4}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{4 \cdot 8}x^{n-4} - \dots,$$

are eigenfunctions of the operator F , and the coefficients in this expression whose parity is different from that of n are zero. The coefficients whose parity is the same as that of n are found by the formula

$$a_{k-2} = \frac{k(k-1)}{2k-2n-4} a_k.$$

The functions φ_n are pairwise orthogonal and $F\varphi_n = \lambda_n \varphi_n$, $\lambda_n^4 = 4\pi^2$. The functions φ_n coincide with the Hermite functions except for a multiplicative constant. The latter are obtained by applying the orthogonalization procedure in $L^2(-\infty, \infty)$ to the functions $e^{-x^2/2}, xe^{-x^2/2}, \dots$.

3. Let us find the Fourier transform of some elementary distributions. Let $f(x) = e^{-x^2/2}$. Taking any line parallel to the real line as a contour of integration and applying the Cauchy residue theorem, we have $F(f) = \sqrt{2\pi}f(\sigma) = \sqrt{2\pi}e^{-\sigma^2/2}$. If $f(x) = 1$, then $F(1) = g(\sigma)$ is determined by the equality

$$\begin{aligned} (F(1), F(\varphi)) &= 2\pi(1, \varphi) = 2\pi \int_{-\infty}^{+\infty} \varphi(x) dx \\ &= 2\pi \int_{-\infty}^{\infty} \varphi(x)e^{ix0} dx = 2\pi g(0) = 2\pi(\delta, g), \end{aligned}$$

i.e., $F(1) = \delta$. Finally, let $f(x) = \delta(x)$. Then

$$(F(\delta), F(\varphi)) = 2\pi(\delta, \varphi) = 2\pi\varphi(0) = \int_{-\infty}^{+\infty} g(\sigma) d\sigma = (1, g),$$

$$\text{i.e., } F(\sigma) = \frac{1}{2\pi}.$$

4. We denote by $P(\xi)$ a polynomial in the variables $\xi_1, \xi_2, \dots, \xi_n$. Let $P(D)$ be the linear differential operator obtained by replacing ξ_j by the operator $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$. The operator $P(D)$ is thus representable in the form

$$P(D) = \sum_{m \geq |\alpha| \geq 0} a_\alpha D^\alpha, \quad D^\alpha = \prod_{j=1}^n D^{\alpha_j}, \quad D^0 = E, \quad D^{\alpha_j} = \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}},$$

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, and $1 \leq \alpha_j \leq m$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$.

A *fundamental solution* corresponding to $P(D)$ is defined to be a distribution \mathcal{E} on \mathbf{R}^n such that $P(D)\mathcal{E} = \delta(x)$.

The expression $u = \mathcal{E} * f$, where f is such that the convolution exists, is a solution of the equation

$$P(D)u = f.$$

Indeed according to the rule for differentiating a convolution, if $u = \mathcal{E} * f$, we have

$$P(D)u = P(D)(\mathcal{E} * f) = P(D)\mathcal{E} * f = \delta * f = f.$$

If $P(D)$ is the Laplacian, $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ in the space \mathbf{R}^n , $n \geq 3$, then

$$\mathcal{E}_n = \frac{1}{(2-n)S_n} |x|^{2-n},$$

where S_n is the area of the unit sphere in \mathbf{R}^n , is a fundamental solution for the operator Δ .

